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# Extrema of Young's modulus for elastic solids with tetragonal symmetry

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**Abstract**

For a homogeneous and linearly elastic solid the general expression of Young's modulus $E(\boldsymbol{n})$ is given, and a constrained extremum problem is formulated for the evaluation of the directions $\boldsymbol{n}$ corresponding to stationary values of the modulus. The formulation follows that presented in [International Journal of Solids and Structures 40 (2003) 1713–1744] for the cubic and transversely isotropic elastic symmetries. In this paper the tetragonal elastic symmetry class is considered, and explicit solutions for the directions $\boldsymbol{n}$ associated to critical points of $E(\boldsymbol{n})$ are analytically evaluated. Properties of these directions and of the corresponding values of the modulus are discussed in detail. The results are presented in terms of three material parameters, which are responsible of the degree of anisotropy. For the tetragonal system, the complete description of the directional dependence of Young's modulus leads to the identification of 12 classes of behavior. For each of these classes several examples of real materials are shown and suitable graphical representations of the function $E(\boldsymbol{n})$ are given as well.


*Keywords:* Anisotropic elasticity; Young's modulus; Tetragonal symmetry

## 1. Introduction

Elastic anisotropy is a common feature of real materials, although engineering materials are usually modelled as macroscopically isotropic. However, being nowadays of growing interest the microstructural aspects of solids and the design of man-made materials produced in order to accomplish specific mechanical

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requirements, the study of elastic anisotropies represents one of the mechanical topics widely studied in recent years.

With the exception of the complete anisotropy (triclinic system) the elastic anisotropy is always restricted by symmetry considerations, which follow from the symmetry elements of the material structure (Cowin and Mehrabadi, 1995). Symmetry considerations are then of paramount importance in the study of the directionality of material properties. Being the macroscopic behavior of a solid strongly conditioned by its microstructural properties, for most materials the basic form of structural symmetry is that contained in the crystal structure. The effects of crystal symmetry in the elastic properties are reported, for instance, in Nye (1957) and Ting (1996). These elastic properties are specified by all the independent elements of the elasticity tensor $\mathbb{C}$, whose number ranges from 3 (cubic system) to 21 (triclinic system). The macroscopic behavior of a solid is then related to its anisotropic properties and in some materials the degree of anisotropy is decidedly non-negligible, leading sometimes to the proximity of material instability. In this work, the directional dependence of Young's modulus is investigated with reference to the tetragonal elastic symmetry, characterized by six elastic constants. The directional dependence of Young's modulus in two dimensions has been previously studied by Goens (1933) and Wooster (1949); some three-dimensional pictures of plaster models of the surface generated by Young's modulus are given by Schmid and Boas (1935) and some analytical studies are provided by Hayes and Shuvalov (1988) and Boulanger and Hayes (1995) for the cubic case. A complete theoretical investigation of Young's modulus for cubic and transversely isotropic solids, a classification of the various cases and the correspondence with real materials are given in Cazzani and Rovati (2003), to which the reader is referred for a detailed general formulation and for a complete list of references. On the basis of the approach given in that paper, the present work theoretically investigates the elastic response of tetragonal solids, to deduce a rational classification in terms of Young's modulus and to recognize the correspondence of the various categories obtained with real materials.

In this section, the problem is formulated in the most general form as a constrained stationarity problem for the evaluation of those directions $\boldsymbol{n}$ along which Young's modulus $E(\boldsymbol{n})$ attains stationary values. The modulus $E(\boldsymbol{n})$ is then a function of the components of the unit vector $\boldsymbol{n}$ and of the Cartesian components of the fourth order elasticity tensor. As formulated here, the problem is equivalent to that formulated by Ostrowska-Maciejewska and Rychlewski (2001), where the aim is to find the extrema of the stored elastic energy for solids under uniaxial tension.

In Section 2 the problem is specialized to the tetragonal elastic symmetry: the stationary points are evaluated, together with the conditions for the existence of such points, in terms of three material parameters $\alpha_2$, $\beta_2$ and $\beta_3$ responsible of the degree of anisotropy (for other definitions of anisotropy parameters, see Nadeau and Ferrari, 2001). The usual Voigt's contracted representation of stress, strain and elasticity tensors is adopted.

In Section 3, a complete classification of the behavior of the function $E(\boldsymbol{n})$ is provided; in particular 12 classes of mechanical response are identified and studied in detail. All the results are given in terms of the material parameters $\alpha_2$, $\beta_2$ and $\beta_3$ (responsible of the discrepancy from isotropy) or of their dimensionless counterparts $A'$, $B'$ and $C'$. It is shown that at each class corresponds at least one real material. For these materials suitable spherical polar diagrams are provided in order to show the directional dependence of the function $E(\boldsymbol{n})$ for each class. Information and data for tetragonal real materials used in this work are taken from Landolt and Börnstein (1992).

A linearly elastic, homogeneous and anisotropic solid, with positive definite stored energy, is considered. The anisotropic elastic character of the material is obviously reflected on Young's modulus, $E$, which is, therefore, a function of direction in the solid. The body is subjected to a unit dipole acting in the direction defined by the unit vector $\boldsymbol{n}$. The problem considered here consists in the evaluation of the directions $\boldsymbol{n}$ corresponding to critical points of the function $E = E(\boldsymbol{n})$. The stress field corresponding to the unit dipole, in absence of body forces, is given by

$$\boldsymbol{\sigma} = \boldsymbol{n} \otimes \boldsymbol{n}. \tag{1}$$

Denoting with $\mathbb{S}$ the positive definite fourth-order compliance tensor, the Hooke's law furnishes the corresponding strain field:

$$\boldsymbol{\epsilon} = \mathbb{S}[\boldsymbol{\sigma}] = \mathbb{S}[\boldsymbol{n} \otimes \boldsymbol{n}]. \tag{2}$$

In view of characterizing the relationship which links the stress and the strain fields, in the direction $\boldsymbol{n}$, the strain tensor (2) is projected along that direction. The expression which defines Young's modulus as a function of the direction $\boldsymbol{n}$ follows immediately:

$$\epsilon(\boldsymbol{n}) = \frac{1}{E(\boldsymbol{n})} = \boldsymbol{n} \otimes \boldsymbol{n} \cdot \mathbb{S}[\boldsymbol{n} \otimes \boldsymbol{n}]. \tag{3}$$

In a Cartesian orthogonal reference frame $Ox_1x_2x_3$, expression (3) can be written in index form as

$$\frac{1}{E(\boldsymbol{n})} = S_{ijhk} n_i n_j n_h n_k, \tag{4}$$

where indices $i$, $j$, $h$, $k$ range from 1 to 3 and the usual rule of sum over a repeated subscript is assumed.

In order to evaluate the direction $\boldsymbol{n}$ for which the modulus $E(\boldsymbol{n})$—or its reciprocal $1/E(\boldsymbol{n})$—attains extreme values, the following Lagrangian function is defined:

$$\mathscr{L}(\boldsymbol{n}, \lambda) = \boldsymbol{n} \otimes \boldsymbol{n} \cdot \mathbb{S}[\boldsymbol{n} \otimes \boldsymbol{n}] + \lambda(\boldsymbol{n} \cdot \boldsymbol{n} - 1), \tag{5}$$

where $\lambda$ is a Lagrangian multiplier associated to the constraint $\boldsymbol{n} \cdot \boldsymbol{n} = 1$.

The stationarity conditions for the Lagrangian function $\mathscr{L}$ are thus

$$\begin{cases} \dfrac{\partial \mathscr{L}(\boldsymbol{n}, \lambda)}{\partial \boldsymbol{n}} = 0, \\ \dfrac{\partial \mathscr{L}(\boldsymbol{n}, \lambda)}{\partial \lambda} = 0, \end{cases} \tag{6}$$

and can be explicitly written, making use of the symmetries on $\mathbb{S}$, as

$$\begin{cases} 2S_{ijhk} n_j n_h n_k + \lambda n_i = 0, \\ n_i n_i - 1 = 0. \end{cases} \tag{7}$$

## 2. Evaluation of the stationary points and conditions for existence

The tetragonal symmetry is characterized by five planes of elastic mirror symmetry, viz.

1. $\Pi_{\mathrm{I}} := x_1 = 0$;
2. $\Pi_{\mathrm{II}} := x_2 = 0$;
3. $\Pi_{\mathrm{III}} := x_3 = 0$;
4. $\Pi_{\mathrm{IV}} := x_1 - x_2 = 0$;
5. $\Pi_{\mathrm{V}} := x_1 + x_2 = 0$.

Four of these planes are orthogonal to the fifth one (i.e. $\Pi_{\mathrm{III}}$) and make angles of $\pi/4$ with respect to one another (see Fig. 1).

The number of elasticities characterizing this symmetry is six. For the tetragonal symmetry, the matrix representation of the elasticity tensor, written in the material reference system, and taking into account the minor and major symmetries on $\mathbb{S}$, can be written as

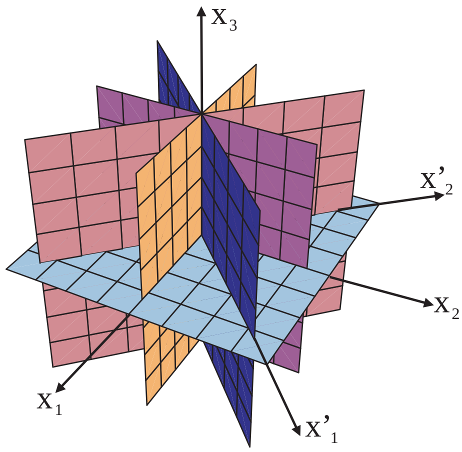


Fig. 1. Planes of elastic mirror symmetry for the tetragonal system.

$$\begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{1122} & S_{1111} & S_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{1133} & S_{1133} & S_{3333} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{1212} & 0 & 0 & S_{1212} \\ 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{1212} & 0 & 0 & S_{1212} \end{pmatrix}. \tag{8}$$

The matrix representation of Voigt's reduced compliances (whose entries are defined as $s_{11} = S_{1111}$, $s_{33} = S_{3333}$, $s_{44} = 4S_{2323}$, $s_{66} = 4S_{1212}$, $s_{12} = S_{1122}$, $s_{13} = S_{1133}$) expressed in the reference system of material symmetry—the system $Ox_1x_2x_3$ in Fig. 1—takes the form:

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{pmatrix}. \tag{9}$$

The assumed existence of a positive definite elastic energy imposes some restrictions on the reduced elastic coefficients in matrix (9): the application of Jordan's lemma to (9) leads to the following constraints on the reduced compliances:

$$s_{11} > 0, \quad s_{33} > 0, \quad s_{44} > 0, \quad s_{66} > 0, \tag{10}$$

$$-s_{11} < s_{12} < s_{11}, \tag{11}$$

$$-\frac{1}{\sqrt{2}}\sqrt{s_{11}s_{33}}\sqrt{1+\frac{s_{12}}{s_{11}}} < s_{13} < \frac{1}{\sqrt{2}}\sqrt{s_{11}s_{33}}\sqrt{1+\frac{s_{12}}{s_{11}}}. \tag{12}$$

In terms of reduced compliances $s_{ij}$ and of the components of the unit vector $\boldsymbol{n}$, expression (4) for $1/E(\boldsymbol{n})$ explicitly reads

$$\frac{1}{E(\boldsymbol{n})} = s_{11} - (s_{11}-s_{33})n_3^4 - (2s_{11}-2s_{12}-s_{66})n_1^2n_2^2 - (2s_{11}-2s_{13}-s_{44})\left(n_1^2n_3^2+n_2^2n_3^2\right), \tag{13}$$

which can be rewritten in the equivalent form:

$$\frac{1}{E(\boldsymbol{n})} = s_{11} - \alpha_2 n_3^4 - \beta_3 n_1^2n_2^2 - \beta_2\left(n_1^2n_3^2+n_2^2n_3^2\right), \tag{14}$$

where the following material parameters have been defined:

$$\alpha_2 := s_{11} - s_{33}, \tag{15}$$

$$\beta_2 := 2s_{11} - 2s_{13} - s_{44}, \tag{16}$$

$$\beta_3 := 2s_{11} - 2s_{12} - s_{66}. \tag{17}$$

It must be noticed that:

1. expression (14) depends on all six independent elastic coefficients;
2. expression (14) differs from the analogous expression for the hexagonal case (see Cazzani and Rovati, 2003) for the presence of the material parameter $\beta_3$, i.e.:

$$\left(\frac{1}{E}\right)_{\text{tetra}} = \left(\frac{1}{E}\right)_{\text{hexa}} - \beta_3 n_1^2n_2^2. \tag{18}$$

Note that the material parameter $\alpha_2$, although given by the difference of two strictly positive material parameters, is not sign-restricted. Anyway, these bounds for $\alpha_2$ are easily found:

$$-s_{33} < \alpha_2 < s_{11}, \tag{19}$$

where the lower and upper bounds are approached in the limit as $s_{11}$ and $s_{33}$ go to zero, respectively.

It must be observed that also the parameters $\beta_2$ and $\beta_3$ are not sign-restricted. In particular, no bounds can be assigned to $\beta_2$: indeed, by its definition (16), it depends on the compliance $s_{13}$ which, in turn, is loosely restricted by $s_{33}$ and $s_{12}$ (which are independent of each other) through inequality (12).

Concerning the parameter $\beta_3$, the bounds $(10)_1$, $(10)_4$ and (11) imply that

$$-s_{66} < \beta_3 < 4s_{11}. \tag{20}$$

In this case, the lower bound is attained for any $s_{11}$ when $s_{12} \to +s_{11}$; the upper bound is reached when, simultaneously, $s_{12} \to -s_{11}$ and $s_{66} \to 0$.

For the elastic symmetry under consideration, the Lagrangian function (5) can be written in the form:

$$\mathscr{L} = s_{11} - \alpha_2 n_3^4 - \beta_3 n_1^2n_2^2 - \beta_2(n_1^2n_3^2+n_2^2n_3^2) + \lambda(n_1^2+n_2^2+n_3^2-1) \tag{21}$$

and the corresponding explicit stationarity conditions (7) read

$$\begin{cases} (-\beta_3 n_2^2 - \beta_2 n_3^2 + \lambda)n_1 = 0, \\ (-\beta_3 n_1^2 - \beta_2 n_3^2 + \lambda)n_2 = 0, \\ (-2\alpha_2 n_3^2 - \beta_2(n_1^2+n_2^2) + \lambda)n_3 = 0, \\ n_1^2+n_2^2+n_3^2 = 1. \end{cases} \tag{22}$$

It is now possible to carry out some assumptions on the solutions: these allow to distinguish three different cases. Case I occurs when the unit vector $\boldsymbol{n}$ has only one non-vanishing Cartesian component; Case II, when $\boldsymbol{n}$ shows two components different from zero and Case III when $\boldsymbol{n}$ has all (i.e. three) non-vanishing components. These three cases will be studied in detail in the following subsections.

### 2.1. Case I: *n* has only one non-vanishing component

First, if $\boldsymbol{n}$ has only one non-vanishing component, i.e., if $n = \pm\boldsymbol{e}_1$, or $\boldsymbol{n} = \pm\boldsymbol{e}_2$, or $\boldsymbol{n} = \pm\boldsymbol{e}_3$, ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ being the unit vectors in the positive direction of the Cartesian reference axes) the following solutions are respectively obtained:

$$(1)\quad \begin{cases} n_1^2 = 1, \\ n_2^2 = n_3^2 = 0, \\ \lambda = 0, \\ \frac{1}{E} = s_{11}, \end{cases} \qquad (2)\quad \begin{cases} n_2^2 = 1, \\ n_1^2 = n_3^2 = 0, \\ \lambda = 0, \\ \frac{1}{E} = s_{11}, \end{cases} \qquad (3)\quad \begin{cases} n_3^2 = 1, \\ n_1^2 = n_2^2 = 0, \\ \lambda = 2\alpha_2, \\ \frac{1}{E} = s_{11} - \alpha_2 = s_{33}. \end{cases} \tag{23}$$

In these solutions the admissibility conditions $0 \leqslant n_i^2 \leqslant 1$ $(i = 1, 2, 3)$ are automatically satisfied, and the positivity condition for Young's modulus is a priori guaranteed in cases (1) and (2), whereas case (3) requires that $\alpha_2 < s_{11}$.

### 2.2. Case II: *n* has two non-vanishing components

The other cases to be considered are those when the unit vector $\boldsymbol{n}$ shows at the same time two non-vanishing components, i.e., if $\boldsymbol{n} = n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$, or $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_3\boldsymbol{e}_3$, or $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2$. In the first case ($\boldsymbol{n} = n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$), the following solution is obtained:

$$(4)\quad \begin{cases} n_1^2 = 0, \\ n_2^2 = 1 - n_3^2 = 1 - \dfrac{\beta_2}{2(\beta_2 - \alpha_2)} = \dfrac{1}{2} - \dfrac{\alpha_2}{2(\beta_2 - \alpha_2)}, \\ n_3^2 = \dfrac{\beta_2}{2(\beta_2 - \alpha_2)} = \dfrac{1}{2} + \dfrac{\alpha_2}{2(\beta_2 - \alpha_2)}, \\ \lambda = \beta_2 n_3^2 = \dfrac{\beta_2^2}{2(\beta_2 - \alpha_2)}, \\ \frac{1}{E} = s_{11} - (\alpha_2 n_3^2 + \beta_2 n_2^2)n_3^2 = s_{11} - \dfrac{\beta_2^2}{4(\beta_2 - \alpha_2)}. \end{cases} \tag{24}$$

This result is the same as in the transversely isotropic case (see Cazzani and Rovati, 2003), because the solution does not depend on the material parameter $\beta_3$ which makes the difference between the hexagonal and the tetragonal cases. In other words, on the coordinate plane $\Pi_\text{I}$ the behavior of Young's modulus is the same for the transverse isotropy and the tetragonal class. By virtue of this, reference can be made to Cazzani and Rovati (2003): conditions ensuring that $n_2$ and $n_3$ be positive, and, moreover, that a positive value of $1/E$ is attained at the stationary point are given in Table 1, where the new parameter $\beta_2^\star$ is defined as

$$\beta_2^\star := 2s_{11}\left(1 + \sqrt{1 - \frac{\alpha_2}{s_{11}}}\right) = 2s_{11}\left(1 + \sqrt{\frac{s_{33}}{s_{11}}}\right).$$

If the second possible form of the unit vector is considered ($\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_3\boldsymbol{e}_3$), the corresponding solution reads

Table 1
Admissibility conditions for the solution of Case II-(4)

| | | | |
|---|---|---|---|
| $\beta_2 \succ \alpha_2$ | $\alpha_2 \succ 0$ | $0 < n_2^2, n_3^2 < 1$ | $\frac{1}{E} > 0$ |
| $\beta_2 > \alpha_2$ | $\alpha_2 > 0$ | $\beta_2 > 2\alpha_2$ | $2\alpha_2 < \beta_2 < \beta_2^\star$ |
| | $\alpha_2 \leqslant 0$ | $\beta_2 \geqslant 0$ | $0 < \beta_2 < \beta_2^\star$ |
| $\beta_2 < \alpha_2$ | $\alpha_2 \geqslant 0$ | $\beta_2 < 0$ | – |
| | $\alpha_2 < 0$ | $\beta_2 < 2\alpha_2$ | – |

$$(5) \quad \begin{cases} n_1^2 = 1 - n_3^2 = 1 - \dfrac{\beta_2}{2(\beta_2 - \alpha_2)} = \dfrac{1}{2} - \dfrac{\alpha_2}{2(\beta_2 - \alpha_2)}, \\ n_2^2 = 0, \\ n_3^2 = \dfrac{\beta_2}{2(\beta_2 - \alpha_2)} = \dfrac{1}{2} + \dfrac{\alpha_2}{2(\beta_2 - \alpha_2)}, \\ \lambda = \beta_2 n_3^2 = \dfrac{\beta_2^2}{2(\beta_2 - \alpha_2)}, \\ \frac{1}{E} = s_{11} - (\alpha_2 n_3^2 + \beta_2 n_1^2) n_3^2 = s_{11} - \dfrac{\beta_2^2}{4(\beta_2 - \alpha_2)}. \end{cases} \tag{25}$$

This case reproduces the previous one with the exchange in roles between $n_1$ and $n_2$, exactly as it appears in the transversely isotropic case. Note again that the behavior of $1/E$ on the plane $\Pi_{\mathrm{II}}$ is independent on the parameter $\beta_3$.

Finally, when $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2$, the solution is

$$(6) \quad \begin{cases} n_1^2 = \frac{1}{2}, \\ n_2^2 = \frac{1}{2}, \\ n_3^2 = 0, \\ \lambda = \frac{\beta_3}{2}, \\ \frac{1}{E} = s_{11} - \frac{\beta_3}{4}. \end{cases} \tag{26}$$

In this case, the value of $1/E$ depends on the material parameter $\beta_3$ and the stationary points belong to the bisectors of the coordinate axes.

The positivity of $n_1$ and $n_2$ is automatically satisfied, whereas the positivity of $1/E$ needs, by virtue of (17), that the following inequality holds:

$$s_{11} - \frac{\beta_3}{4} > 0 \quad \Rightarrow \quad \beta_3 < 4s_{11}. \tag{27}$$

Now, by the definition of $\beta_3$, (17), it follows:

$$s_{11} > -s_{12} - \frac{s_{66}}{2}, \tag{28}$$

which is always fulfilled for any admissible value of the elastic coefficient $s_{66}$. Indeed, the upper bound given by (20) guarantees that $1/E > 0$; if, on the other hand, $\beta_3$ is seen as an independent variable, $(27)_2$ furnishes an upper bound corresponding to (20).

It is useful to notice that the above-found stationary point is a maximum when

$$\frac{1}{E} > s_{11} \quad \Rightarrow \quad \beta_3 < 0, \quad \text{that is, if } 2(s_{11} - s_{12}) < s_{66}, \tag{29}$$

whereas it is a minimum if

$$\frac{1}{E} < s_{11} \quad \Rightarrow \quad \beta_3 > 0, \quad \text{that is, if } 2(s_{11} - s_{12}) > s_{66}. \tag{30}$$

These two conditions allow to define two classes of behavior characterized by a shear stiffness respectively lower, (29), or higher, (30), compared to that of an—at least transversely—isotropic material.

### 2.3. Case III: $\boldsymbol{n}$ has all non-vanishing components

The last case to be considered is that of $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2 e_2 + n_3\boldsymbol{e}_3$:

$$(7) \quad \begin{cases} n_1^2 = \dfrac{\beta_2 - 2\alpha_2}{4(\beta_2 - \alpha_2) - \beta_3}, \\ n_2^2 = \dfrac{\beta_2 - 2\alpha_2}{4(\beta_2 - \alpha_2) - \beta_3}, \\ n_3^2 = \dfrac{2\beta_2 - \beta_3}{4(\beta_2 - \alpha_2) - \beta_3}, \\ \lambda = \dfrac{2(\beta_2^2 - \alpha_2\beta_3)}{4(\beta_2 - \alpha_2) - \beta_3}, \\ \frac{1}{E} = s_{11} - \dfrac{\beta_2^2 - \alpha_2\beta_3}{4(\beta_2 - \alpha_2) - \beta_3}. \end{cases} \tag{31}$$

First, it must be noticed that this stationary point shows equal values of $n_1^2$ and $n_2^2$, so that it belongs to one of the planes that bisect the coordinate axes $x_1$ and $x_2$, i.e. the planes $\Pi_{\text{IV}}$ or $\Pi_{\text{V}}$. The value of $1/E$ is different from those found in the previous three cases, because on one hand there is an explicit dependence on $\beta_3$—which does not appear in Cases II-(4) and II-(5)—and, on the other one, the material parameters $\alpha_2$ and $\beta_2$ appear explicitly in the definition of $1/E$, differently from Case II-(6).

In order for the solution to be acceptable, the admissibility conditions $n_1^2 = n_2^2 > 0$ and $n_3^2 > 0$ must be fulfilled. These require that

$$\begin{cases} 0 < \dfrac{\beta_2 - 2\alpha_2}{4(\beta_2 - \alpha_2) - \beta_3}, \\ 0 < \dfrac{2\beta_2 - \beta_3}{4(\beta_2 - \alpha_2) - \beta_3}. \end{cases} \tag{32}$$

Two cases must be considered.

1. If $\beta_2 > \alpha_2 + \beta_3/4$, (32) imply

$$\begin{cases} \beta_2 > 2\alpha_2 \\ \beta_2 > \frac{\beta_3}{2} \end{cases} \quad \Rightarrow \quad \beta_2 > \max\left(2\alpha_2, \frac{\beta_3}{2}\right). \tag{33}$$

2. If $\beta_2 < \alpha_2 + \beta_3/4$, (32) imply

$$\begin{cases} \beta_2 < 2\alpha_2 \\ \beta_2 < \frac{\beta_3}{2} \end{cases} \quad \Rightarrow \quad \beta_2 < \min\left(2\alpha_2, \frac{\beta_3}{2}\right). \tag{34}$$

Once ensured that $0 < n_i^2$ (for $i = 1, 2, 3$), it is easily verified that

$$n_1^2 + n_2^2 + n_3^2 = \frac{2(\beta_2 - 2\alpha_2) + (2\beta_2 - \beta_3)}{4(\beta_2 - \alpha_2) - \beta_3} = 1.$$

It is now necessary to enforce the condition $1/E > 0$; it implies

$$s_{11} - \frac{\beta_2^2 - \alpha_2\beta_3}{4(\beta_2 - \alpha_2) - \beta_3} > 0. \tag{35}$$

Again, two cases have to be considered, according to the sign of the denominator of (35):

1. If $\beta_2 > \alpha_2 + \beta_3/4$, it must result

$$\begin{cases} -\beta_2^2 + 4s_{11}\beta_2 + [\alpha_2\beta_3 - s_{11}(4\alpha_2 + \beta_3)] > 0, \\ \beta_2 > \alpha_2 + \beta_3/4. \end{cases} \tag{36}$$

The left hand of inequality $(36)_1$ is a quadratic function of $\beta_2$, and is therefore satisfied only within the range defined by the roots (which must be real and distinct) of the associated quadratic algebraic equation:

$$\beta_2^2 - 4s_{11}\beta_2 - [\alpha_2\beta_3 - s_{11}(4\alpha_2 + \beta_3)] = 0. \tag{37}$$

The roots of (37) written in increasing order are

$$\beta_2^{\mathrm{I}} := 2s_{11} - \sqrt{(4s_{11} - \beta_3)(s_{11} - \alpha_2)}, \tag{38}$$

$$\beta_2^{\mathrm{II}} := 2s_{11} + \sqrt{(4s_{11} - \beta_3)(s_{11} - \alpha_2)}, \tag{39}$$

and turn out to be real and distinct if and only if the discriminant is positive, i.e.:

$$(4s_{11} - \beta_3)(s_{11} - \alpha_2) > 0. \tag{40}$$

However the bounds (19), (20) cannot be violated, so that the only acceptable solution is

$$\begin{cases} 4s_{11} - \beta_3 > 0 \\ s_{11} - \alpha_2 > 0 \end{cases} \Rightarrow \begin{cases} \beta_3 < 4s_{11} \\ \alpha_2 < s_{11}. \end{cases} \tag{41}$$

The following result is therefore obtained:

$$\begin{cases} \beta_2 > \alpha_2 + \beta_3/4, \\ \beta_2^{\mathrm{I}} < \beta_2 < \beta_2^{\mathrm{II}} \end{cases} \tag{42}$$

with $\beta_3 < 4s_{11}$, $\alpha_2 < s_{11}$. It can be easily checked that the strongest inequality resulting from (42) is simply:

$$\beta_2^{\mathrm{I}} < \beta_2 < \beta_2^{\mathrm{II}}.$$

2. If $\beta_2 < \alpha_2 + \beta_3/4$, it must result

$$\begin{cases} -\beta_2^2 + 4s_{11}\beta_2 + [\alpha_2\beta_3 - s_{11}(4\alpha_2 + \beta_3)] < 0, \\ \beta_2 < \alpha_2 + \beta_3/4. \end{cases} \tag{43}$$

The left hand of inequality (43) is now a quadratic function of $\beta_2$, which can be satisfied only outside the range of the roots (38), (39) of the associated quadratic algebraic equation (37), which must be again real and distinct if bounds (19), (20) cannot be violated.As a consequence it must result

$$\begin{cases} \beta_2 < \alpha_2 + \beta_3/4, \\ \beta_2 < \beta_2^{\mathrm{I}} \cup \beta_2 > \beta_2^{\mathrm{II}} \end{cases} \tag{44}$$

with $\beta_3 < 4s_{11}$, $\alpha_2 < s_{11}$. It can be easily checked again that the strongest inequality resulting from (44) is simply:

Table 2
Admissibility conditions for the solution of Case III-(7)

| | | |
|---|---|---|
| $\beta_2 \succ \frac{\beta_3}{4} + \alpha_2$ | $0 < n_1^2, n_2^2, n_3^2 < 1$ | $\frac{1}{E} > 0$ |
| $\beta_2 > \frac{\beta_3}{4} + \alpha_2$ | $\beta_2 > \max(2\alpha_2, \frac{\beta_3}{2})$ | $\max(2\alpha_2, \frac{\beta_3}{2}) < \beta_2 < \beta_2^{\star\star}$ |
| $\beta_2 < \frac{\beta_3}{4} + \alpha_2$ | $\beta_2 < \min(2\alpha_2, \frac{\beta_3}{2})$ | – |

$$\beta_2 < \frac{\beta_3}{4} + \alpha_2,$$

since $\beta_3/4 + \alpha_2 < \beta_2^{\mathrm{I}}$.

Finally, it is straightforward to verify that

$$\min\left(2\alpha_2, \frac{\beta_3}{2}\right) \leqslant \beta_2^{\mathrm{I}} \leqslant \max\left(2\alpha_2, \frac{\beta_3}{2}\right), \tag{45}$$

and recognize that the prescription resulting from the condition $1/E > 0$ can be simply written, in Case 1 above, as

$$\max\left(2\alpha_2, \frac{\beta_3}{2}\right) < \beta_2 < \beta_2^{\mathrm{II}}. \tag{46}$$

An admissible solution for Case III, both in terms of $n_i^2$ ($i = 1, 2, 3$), and of $1/E$ can be synthetically given in Table 2, where the newly introduced parameter $\beta_2^{\star\star}$ is defined as

$$\beta_2^{\star\star} := 2s_{11} + \sqrt{(4s_{11} - \beta_3)(s_{11} - \alpha_2)}.$$

## 3. Stationary values of Young's modulus $E$: classification and examples

Stationary values of Young's modulus reciprocal, $1/E$, have been obtained in the previous Section for tetragonal symmetry.

If symmetries of the tetragonal class are taken advantage of, it is possible to construct the whole surface representing a spherical polar diagram of $1/E(\boldsymbol{n})$ (or of $E(\boldsymbol{n})$) by simply considering one-sixteenth of it, specifically the region of the octant bounded by coordinate planes $\Pi_{\mathrm{II}}$, $\Pi_{\mathrm{III}}$ and by the bisector plane $\Pi_{\mathrm{IV}}$.

When all admissibility conditions are satisfied (both in terms of positive values of $n_i^2$ (with $i = 1, 2, 3$) and of $1/E$) there are in such a region five different stationary points. For the reader's convenience, such points are denoted by their spherical coordinates, namely longitude $\phi$ (measured on coordinate plane $\Pi_{\mathrm{III}}$ starting from plane $\Pi_{\mathrm{II}}$); and latitude, $\vartheta$ (measured starting from coordinate plane $\Pi_{\mathrm{III}}$). These stationary points are as follows:

1. $\frac{1}{E_1} := \frac{1}{E}\Big|_{(0,0)} = s_{11}$.
   It belongs to the coordinate axis $x_1$ and lies on the intersection of planes $\Pi_{\mathrm{II}}$ and $\Pi_{\mathrm{III}}$. (On the whole surface there are 4 points like this one.)
2. $\frac{1}{E_2} := \frac{1}{E}\Big|_{(\cdot,\pi/2)} = s_{11} - \alpha_2$.
   It belongs to the coordinate axis $x_3$, and lies on the intersection of the coordinate plane $\Pi_{\mathrm{II}}$ and the bisector plane $\Pi_{\mathrm{IV}}$. (There are, on the whole surface, 2 points like this.)

3. $\frac{1}{E_3} := \frac{1}{E}\Big|_{(\pi/4,0)} = s_{11} - \frac{\beta_3}{4}$.
   It belongs to the coordinate plane $\Pi_{\text{III}}$, always corresponding to the intersection with the bisector plane $\Pi_{\text{IV}}$. (There are 4 points like this on the whole surface.)
4. $\frac{1}{E_4} := \frac{1}{E}\Big|_{(0,\vartheta)} = s_{11} - \frac{\beta_2^2}{4(\beta_2-\alpha_2)}$.
   It belongs to the coordinate plane $\Pi_{\text{II}}$. (There are 8 of these points on the whole surface.)
5. $\frac{1}{E_5} := \frac{1}{E}\Big|_{(\pi/4,\vartheta)} = s_{11} - \frac{\beta_2^2-\alpha_2\beta_3}{4(\beta_2-\alpha_2)-\beta_3}$.
   It belongs to the bisector plane $\Pi_{\text{IV}}$. (There are 8 points like this on the whole surface.)

It should be noticed that the stationary points $1/E_1$, $1/E_2$ and $1/E_3$ are always present; the last two, namely $1/E_4$ and $1/E_5$, appear only if the conditions reported in Tables 1 and 2, respectively, are satisfied.

In order to characterize the stationary points, it is necessary to understand that the name *minimum* or *maximum*, refers always to a *constrained minimum/maximum*, i.e. an extremum point belonging to a particular axis or plane. Moreover, stationary points are relevant to the radial vector describing in spherical coordinates the surface $1/E(\boldsymbol{n})$.

With reference to $1/E_2$ and $1/E_3$ it can be stated that:

- $\frac{1}{E_2}$ is a *minimum* value on both planes $\Pi_{\text{II}}$ and $\Pi_{\text{IV}}$ whenever $\alpha_2 > 0$;
- $\frac{1}{E_2}$ is a *maximum* value on both planes $\Pi_{\text{II}}$ and $\Pi_{\text{IV}}$ whenever $\alpha_2 < 0$;
- $\frac{1}{E_3}$ is a *minimum* value on the plane $\Pi_{\text{III}}$ whenever $\beta_3 > 0$;
- $\frac{1}{E_3}$ is a *maximum* value on the plane $\Pi_{\text{III}}$ whenever $\beta_3 < 0$.

Moreover, on the bisector plane $\Pi_{\text{IV}}$ it results

$$\frac{1}{E_3} > \frac{1}{E_2} \quad \text{when } \alpha_2 > \frac{\beta_3}{4},$$

$$\frac{1}{E_3} < \frac{1}{E_2} \quad \text{when } \alpha_2 < \frac{\beta_3}{4}.$$

It suffices then to investigate:

1. what is the role of the stationary point $1/E_4$ (which appears *only* when conditions listed in Table 1 are fulfilled) on the coordinate plane $\Pi_{\text{II}}$ with reference to stationary points $1/E_1$ and $1/E_2$;
2. what is the role of the stationary point $1/E_5$ (which appears *only* when conditions listed in Table 2 are fulfilled) on the bisector plane $\Pi_{\text{IV}}$ with reference to stationary points $1/E_3$ and $1/E_2$.

After some lengthy checks, it follows that:

- $\frac{1}{E_4}$ is a *minimum* value on plane $\Pi_{\text{II}}$ when both $\beta_2 > \alpha_2$ and conditions listed in Table 1 are fulfilled;
- $\frac{1}{E_4}$ is a *maximum* value on plane $\Pi_{\text{II}}$ when both $\beta_2 < \alpha_2$ and conditions listed in Table 1 are fulfilled;
- $\frac{1}{E_5}$ is a *minimum* value on plane $\Pi_{\text{IV}}$ when both $\beta_2 > \frac{\beta_3}{4} + \alpha_2$ and conditions listed in Table 2 are fulfilled;
- $\frac{1}{E_5}$ is a *maximum* value on plane $\Pi_{\text{IV}}$ when both $\beta_2 < \frac{\beta_3}{4} + \alpha_2$ and conditions listed in Table 2 are fulfilled.

It is useful noting that on plane $\Pi_{\text{II}}$, whenever $\beta_2 = 0$, the stationary value $1/E_4$ corresponds to $\vartheta = 0$, i.e. it coincides with $1/E_1$; on the other hand, whenever $\beta_2 = 2\alpha_2$, the stationary value $1/E_4$ corresponds to $\vartheta = \pi/2$, coinciding with $1/E_2$. Similarly, on plane $\Pi_{\text{IV}}$, whenever $\beta_2 = \beta_3/2$, the stationary value $1/E_5$ corresponds to $\vartheta = 0$, i.e. it coincides with $1/E_3$; whilst, whenever $\beta_2 = 2\alpha_2$, the stationary value $1/E_5$ corresponds to $\vartheta = \pi/2$, coinciding with $1/E_2$.

Table 3
Classification of stationary points of the surface $1/E(\boldsymbol{n})$ for tetragonal symmetry

| Class | $\alpha_2 \succ 0$ | $\beta_3 \succ 0$ | $\beta_2 \succ \alpha_2$ | $\beta_2 \succ \beta_3/4+\alpha_2$ | $\frac{1}{E_2}$ | $\frac{1}{E_3}$ | $\frac{1}{E_4}$ | $\frac{1}{E_5}$ | Resulting order relationship for $\beta_2$ |
|---|---|---|---|---|---|---|---|---|---|
| I | $\alpha_2>0$ | $\beta_3>0$ | $\beta_2>\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $m$ | $m$ | $m$ | $m$ | $\beta_2>\beta_3/4+\alpha_2$ |
| II | $\alpha_2>0$ | $\beta_3>0$ | $\beta_2<\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $m$ | $m$ | $M$ | $m$ | $\alpha_2+\beta_3/4<\beta_2<\alpha_2$ † |
| III | $\alpha_2>0$ | $\beta_3>0$ | $\beta_2>\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $m$ | $m$ | $m$ | $M$ | $\alpha_2<\beta_2<\beta_3/4+\alpha_2$ |
| IV | $\alpha_2>0$ | $\beta_3>0$ | $\beta_2<\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $m$ | $m$ | $M$ | $M$ | $\beta_2<\alpha_2$ |
| V | $\alpha_2>0$ | $\beta_3<0$ | $\beta_2>\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $m$ | $M$ | $m$ | $m$ | $\beta_2>\alpha_2$ |
| VI | $\alpha_2>0$ | $\beta_3<0$ | $\beta_2<\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $m$ | $M$ | $M$ | $m$ | $\beta_3/4+\alpha_2<\beta_2<\alpha_2$ |
| VII | $\alpha_2>0$ | $\beta_3<0$ | $\beta_2>\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $m$ | $M$ | $m$ | $M$ | $\alpha_2<\beta_2<\beta_3/4+\alpha_2$ † |
| VIII | $\alpha_2>0$ | $\beta_3<0$ | $\beta_2<\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $m$ | $M$ | $M$ | $M$ | $\beta_2<\beta_3/4+\alpha_2$ |
| IX | $\alpha_2<0$ | $\beta_3>0$ | $\beta_2>\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $M$ | $m$ | $m$ | $m$ | $\beta_2>\beta_3/4+\alpha_2$ |
| X | $\alpha_2<0$ | $\beta_3>0$ | $\beta_2<\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $M$ | $m$ | $M$ | $m$ | $\beta_3/4+\alpha_2<\beta_2<\alpha_2$ † |
| XI | $\alpha_2<0$ | $\beta_3>0$ | $\beta_2>\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $M$ | $m$ | $m$ | $M$ | $\alpha_2<\beta_2<\beta_3/4+\alpha_2$ |
| XII | $\alpha_2<0$ | $\beta_3>0$ | $\beta_2<\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $M$ | $m$ | $M$ | $M$ | $\beta_2<\beta_3/4+\alpha_2$ |
| XIII | $\alpha_2<0$ | $\beta_3<0$ | $\beta_2>\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $M$ | $M$ | $m$ | $m$ | $\beta_2>\alpha_2$ |
| XIV | $\alpha_2<0$ | $\beta_3<0$ | $\beta_2<\alpha_2$ | $\beta_2>\beta_3/4+\alpha_2$ | $M$ | $M$ | $M$ | $m$ | $\beta_3/4+\alpha_2<\beta_2<\alpha_2$ |
| XV | $\alpha_2<0$ | $\beta_3<0$ | $\beta_2>\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $M$ | $M$ | $m$ | $M$ | $\alpha_2<\beta_2<\beta_3/4+\alpha_2$ † |
| XVI | $\alpha_2<0$ | $\beta_3<0$ | $\beta_2<\alpha_2$ | $\beta_2<\beta_3/4+\alpha_2$ | $M$ | $M$ | $M$ | $M$ | $\beta_2<\beta_3/4+\alpha_2$ |

These results allow to identify the following distinct classes, defined by all possible combinations of order relationships between the values $\alpha_2$ and 0; $\beta_3$ and 0; $\beta_2$ and $\alpha_2$; $\beta_2$ and $(\beta_3/4+\alpha_2)$, which, on their own, define the nature (i.e. minimum or maximum) of stationary points $1/E_2$, $1/E_3$, $1/E_4$ and $1/E_5$ respectively. It is tacitly assumed that in order to ensure the existence of $1/E_4$ and $1/E_5$ the conditions listed in Tables 1 and 2 must be fulfilled.

If *constrained minima* are denoted by the symbol $m$, *constrained maxima* by the symbol $M$, and the notation † is adopted to outline an order relationship which cannot be satisfied, the 16 classes in Table 3 can be defined.

It should be noticed that four classes, namely II, VII, X and XV, prescribe conflicting attributes to $\beta_2$ and must therefore be removed; as a consequence only 12 admissible classes can exist when characterizing the stationary points of $1/E$ for tetragonal symmetry.

If now attention is turned to the expression of Young's modulus $E$ (and not its reciprocal $1/E$), the previous results can be directly applied, provided that the role of *minima* and *maxima* are exchanged, whereas the reciprocal of the stationary values are considered.

As a matter of fact, if spherical coordinates are again used to describe the surface $E(\boldsymbol{n})$, the five above mentioned stationary points are as follows:

1. $E_1 := E|_{(0,0)} = \left(\frac{1}{E_1}\right)^{-1}$.
   It belongs to the $x_1$ axis, at the intersection of planes $\Pi_{\mathrm{II}}$ and $\Pi_{\mathrm{III}}$.
2. $E_2 := E|_{(\cdot,\pi/2)} = \left(\frac{1}{E_2}\right)^{-1}$.
   It belongs to the $x_3$ axis, at the intersection of planes $\Pi_{\mathrm{II}}$ and $\Pi_{\mathrm{IV}}$.
3. $E_3 := E|_{(\pi/4,0)} = \left(\frac{1}{E_3}\right)^{-1}$.
   It is located on the plane $\Pi_{\mathrm{III}}$, always at the intersection with plane $\Pi_{\mathrm{IV}}$.
4. $E_4 := E|_{(0,\vartheta)} = \left(\frac{1}{E_4}\right)^{-1}$.
   It belongs to the plane $\Pi_{\mathrm{II}}$, provided that conditions listed in Table 1 are fulfilled.
5. $E_5 := E|_{(\pi/4,\vartheta)} = \left(\frac{1}{E_5}\right)^{-1}$.
   It belongs to the bisector plane $\Pi_{\mathrm{IV}}$, provided that conditions listed in Table 2 are fulfilled.

Table 4
Classification of stationary points of the surface $E(\boldsymbol{n})$ for tetragonal symmetry

| Class | $\alpha_2 \succ 0$ | $\beta_3 \succ 0$ | $\beta_2 \succ \alpha_2$ | $\beta_2 \succ \beta_3/4 + \alpha_2$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | Reference figures |
|---|---|---|---|---|---|---|---|---|---|
| 1a | $\alpha_2 > 0$ | $\beta_3 > 0$ | $\beta_2 > \alpha_2$ | $\beta_2 > \beta_3/4 + \alpha_2$ | *M* | *M* | *M* | *M* | 3, 4 |
| 1b | $\alpha_2 > 0$ | $\beta_3 > 0$ | $\beta_2 > \alpha_2$ | $\beta_2 < \beta_3/4 + \alpha_2$ | *M* | *M* | *M* | *m* | 5, 6 |
| 1c | $\alpha_2 > 0$ | $\beta_3 > 0$ | $\beta_2 < \alpha_2$ | $\beta_2 < \beta_3/4 + \alpha_2$ | *M* | *M* | *m* | *m* | 7, 8 |
| 1d | $\alpha_2 > 0$ | $\beta_3 < 0$ | $\beta_2 > \alpha_2$ | $\beta_2 > \beta_3/4 + \alpha_2$ | *M* | *m* | *M* | *M* | 10, 11 |
| 1e | $\alpha_2 > 0$ | $\beta_3 < 0$ | $\beta_2 < \alpha_2$ | $\beta_2 > \beta_3/4 + \alpha_2$ | *M* | *m* | *m* | *M* | 12, 13 |
| 1f | $\alpha_2 > 0$ | $\beta_3 < 0$ | $\beta_2 < \alpha_2$ | $\beta_2 < \beta_3/4 + \alpha_2$ | *M* | *m* | *m* | *m* | 14, 15 |
| 2a | $\alpha_2 < 0$ | $\beta_3 > 0$ | $\beta_2 > \alpha_2$ | $\beta_2 > \beta_3/4 + \alpha_2$ | *m* | *M* | *M* | *M* | 17, 18 |
| 2b | $\alpha_2 < 0$ | $\beta_3 > 0$ | $\beta_2 > \alpha_2$ | $\beta_2 < \beta_3/4 + \alpha_2$ | *m* | *M* | *M* | *m* | 19, 20 |
| 2c | $\alpha_2 < 0$ | $\beta_3 > 0$ | $\beta_2 < \alpha_2$ | $\beta_2 < \beta_3/4 + \alpha_2$ | *m* | *M* | *m* | *m* | 21, 22 |
| 2d | $\alpha_2 < 0$ | $\beta_3 < 0$ | $\beta_2 > \alpha_2$ | $\beta_2 > \beta_3/4 + \alpha_2$ | *m* | *m* | *M* | *M* | 24, 25 |
| 2e | $\alpha_2 < 0$ | $\beta_3 < 0$ | $\beta_2 < \alpha_2$ | $\beta_2 > \beta_3/4 + \alpha_2$ | *m* | *m* | *m* | *M* | 26, 27 |
| 2f | $\alpha_2 < 0$ | $\beta_3 < 0$ | $\beta_2 < \alpha_2$ | $\beta_2 < \beta_3/4 + \alpha_2$ | *m* | *m* | *m* | *m* | 28, 29 |

When the above-listed stationary values of Young's modulus are expressed as functions of the elastic coefficients $s_{11}$, $s_{12}$, $s_{13}$, $s_{33}$, $s_{44}$, $s_{66}$ the following results are obtained:

$$E_1 = \frac{1}{s_{11}}, \tag{47}$$

$$E_2 = \frac{1}{s_{33}}, \tag{48}$$

$$E_3 = \frac{4}{2s_{11} + 2s_{12} + s_{66}}, \tag{49}$$

$$E_4 = \frac{4(s_{11} + s_{33} - 2s_{13} - s_{44})}{4(s_{11}s_{33} - s_{13}^2 - s_{13}s_{44}) - s_{44}^2}, \tag{50}$$

$$E_5 = \frac{2s_{11} + 4s_{33} + 2s_{12} - 8s_{13} - 4s_{44} + s_{66}}{2s_{11}s_{33} - 4s_{13}^2 + 2s_{12}s_{33} - 4s_{13}s_{44} - s_{44}^2 + s_{33}s_{66}}. \tag{51}$$

It is now possible, when conditions listed in Tables 1 and 2 are fulfilled, to identify the 12 distinct classes of material behavior listed in Table 4. It should be emphasized that with the data available in the literature (Landolt and Börnstein, 1992) it has been possible to find for each class at least one material belonging to it.

For each of the 12 outlined classes some parametric plots of the surface $E(\boldsymbol{n})$ have been produced in the range of spherical coordinates $0 \leqslant \phi \leqslant \pi/2 \cup 0 \leqslant \vartheta \leqslant \pi/2$ in order to show the evolution exhibited by the surface as a function of changes of material parameters.

For ease of comparison purposes, these plots have been produced by using the dimensionless counterparts of material parameters $\alpha_2$, $\beta_2$, $\beta_3$, namely $A'$, $B'$ and $C'$, defined as follows:

$$A' := \frac{s_{33}}{s_{11}}, \quad B' := \frac{2s_{13} + s_{44}}{2s_{11}}, \quad C' := \frac{2s_{12} + s_{66}}{2s_{11}}. \tag{52}$$

It can be easily shown that the inverse relations of (52), by virtue of (15)–(17) are simply:

$$\alpha_2 = s_{11}(1 - A'), \quad \beta_2 = 2s_{11}(1 - B'), \quad \beta_3 = 2s_{11}(1 - C'). \tag{53}$$

Moreover, for each class a figure showing the whole surface generated by $E(\boldsymbol{n})$ for some representing materials is given.

### 3.1. Class 1a

It is defined by the following ranges of the material parameters:

$$\alpha_2 > 0; \quad \beta_3 > 0; \quad \beta_2 > \alpha_2 + \beta_3/4,$$

or, in dimensionless form,

$$A' < 1; \quad C' < 1; \quad B' < \frac{1 + C' + 2A'}{4}.$$

As shown in Fig. 2, where extension of contiguous classes 1a, 1b and 1c as a function of material parameter $\beta_2$ are outlined, the existence of the stationary value $E_4$ is restricted to the range:

$$\max(\alpha_2 + \beta_3/4, 2\alpha_2) < \beta_2 < \beta_2^{\star},$$

whereas that of $E_5$ is delimited by

$$\max(\beta_3/2, 2\alpha_2) < \beta_2 < \beta_2^{\star\star}.$$

Since within this class it happens that $\beta_2^{\star\star} < \beta_2^{\star}$, then it follows that *both* stationary values $E_4$ and $E_5$ exist only if

$$\max(\beta_3/2, 2\alpha_2) < \beta_2 < \beta_2^{\star\star}.$$

Fig. 3 shows the evolution of the surface $E(\boldsymbol{n})$ when the material parameters $B'$ and $C'$ do change.

Fig. 4 presents the whole surface generated by the directional dependence of Young's modulus for some representative materials, listed in the figure caption.

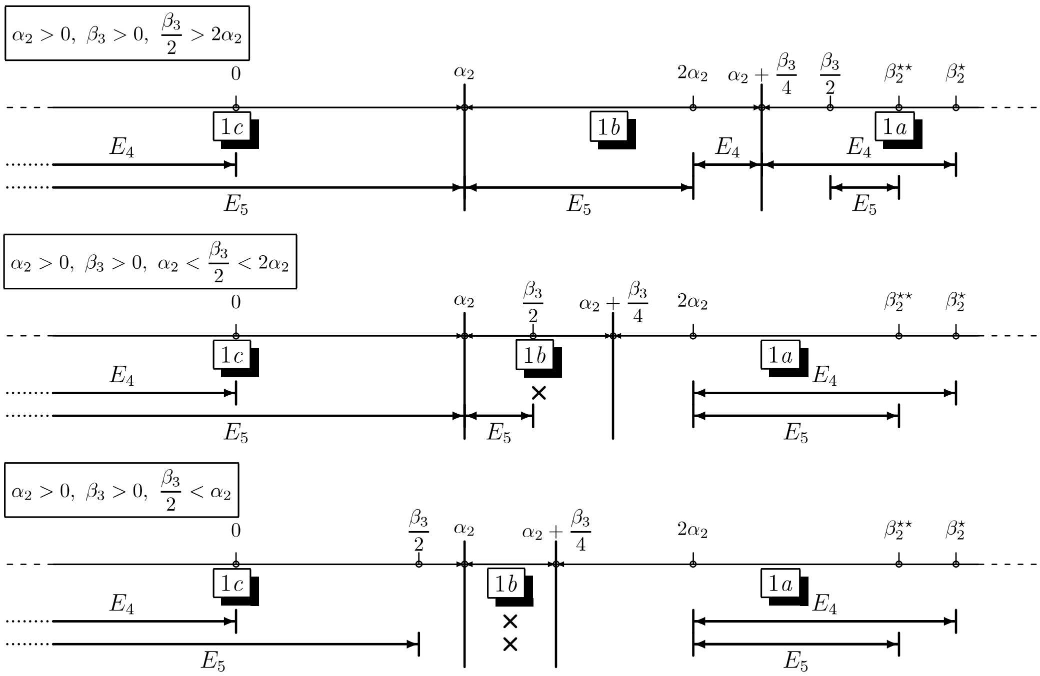


Fig. 2. Extension of Classes 1a, 1b, 1c as a function of the material parameter $\beta_2$ and corresponding ranges of existence of stationary values $E_4$ and $E_5$.

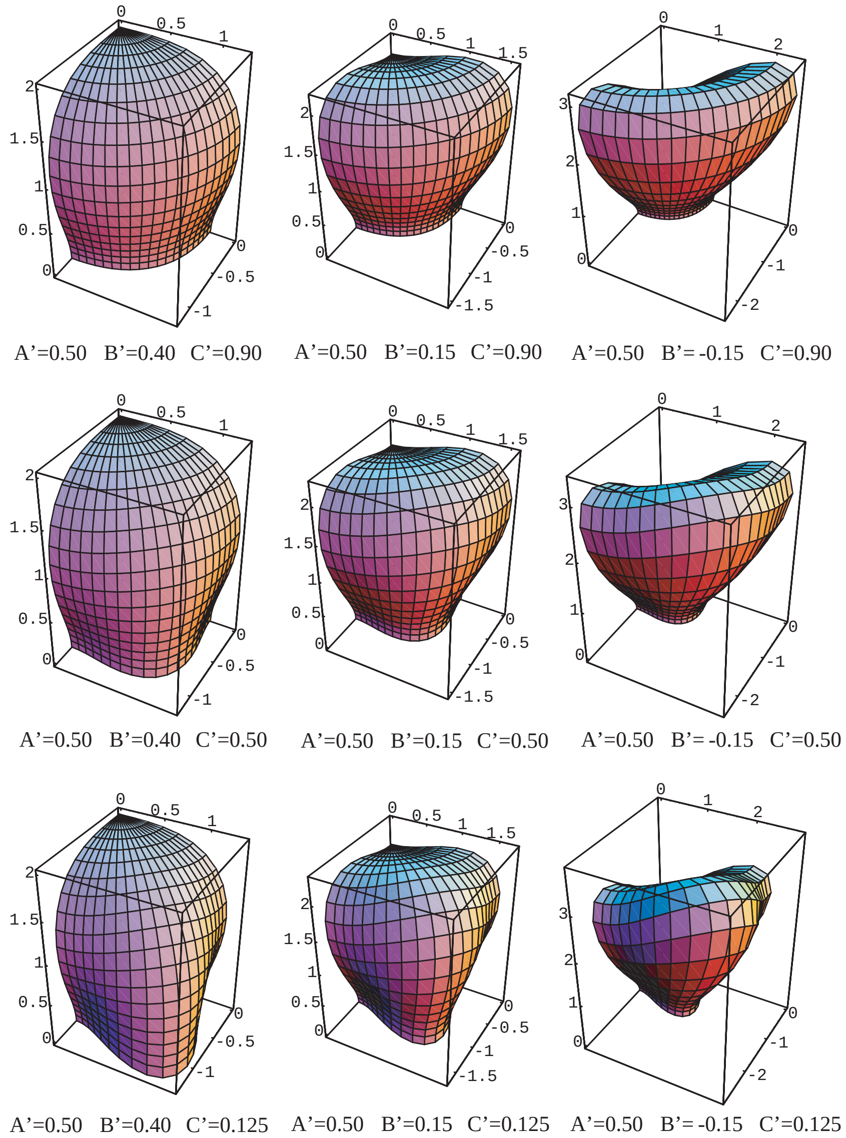


Fig. 3. Class 1a: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

### 3.2. Class 1b

The material parameter ranges are, in this case,

$$\alpha_2 > 0; \quad \beta_3 > 0; \quad \alpha_2 < \beta_2 < \alpha_2 + \beta_3/4,$$

i.e. in dimensionless form:

$$A' < 1; \quad C' < 1; \quad \frac{1 + C' + 2A'}{4} < B' < \frac{1 + A'}{2}.$$

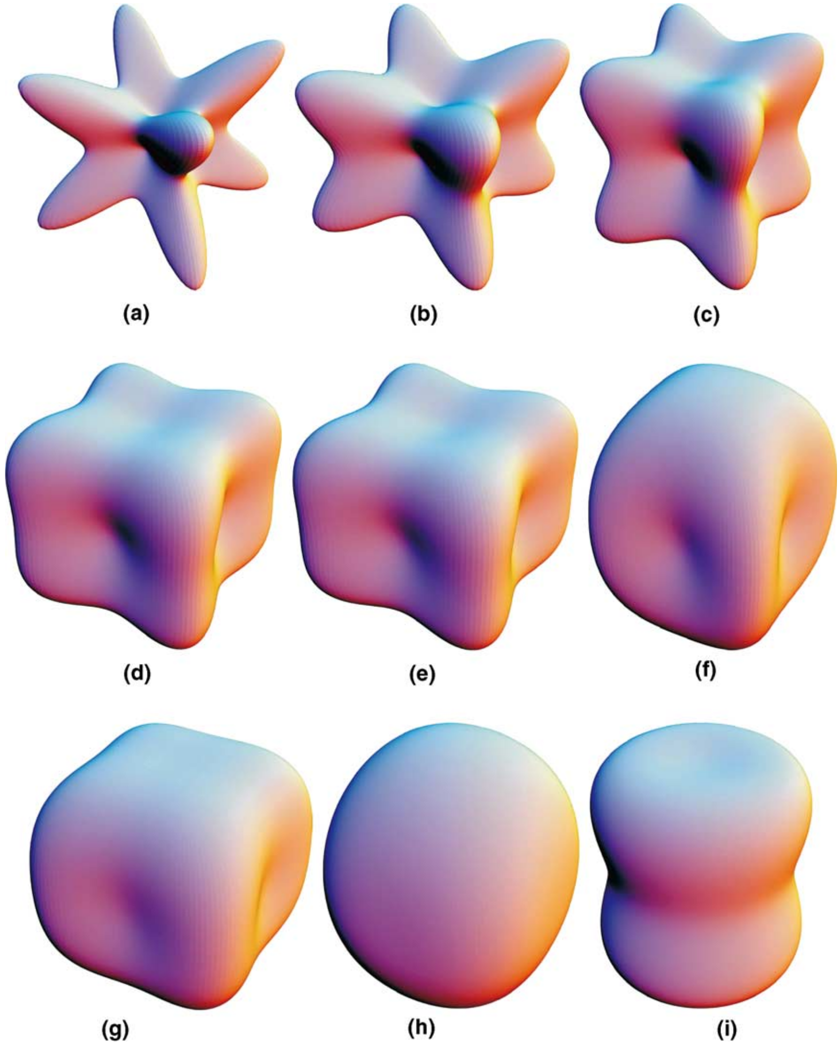


Fig. 4. Tetragonal system: materials belonging to Class 1a, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) In–Tl (indium–thallium, atomic percentage Tl: 15%); (b) indium–cadmium alloy (atomic percentage Cd: 3.4%); (c) indium–lead (atomic percentage Pb: 5%); (d) $HgIn_2Te_4$ (mercury indium telluride, vacancy compound); (e) CoPt (cobalt platinum); (f) stishovite; (g) β-$CdP_2$ (cadmium phosphide, at 100 K); (h) $MoSi_2$ (molybdenum disilicide); (i) TlSe (thallium selenide). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 0.96 | −0.21 | −0.33 | 10.0 | 645.0 | 707.0 | 266.00 | 256.00 | 133.00 | 93.00 | −134.00 | −123.00 | .003760 | .023314 |
| (b) | 0.97 | −0.04 | −0.25 | 4.0 | 364.0 | 436.3 | 175.00 | 171.00 | 147.00 | 89.30 | −87.80 | −80.50 | .005714 | .022356 |
| (c) | 0.49 | 0.08 | −0.22 | 94.0 | 337.0 | 448.0 | 183.00 | 89.00 | 107.00 | 192.00 | −137.00 | −39.00 | .005464 | .021445 |
| (d) | 0.84 | 0.34 | 0.08 | 6.4 | 51.5 | 71.5 | 38.80 | 32.40 | 46.700 | 41.50 | −17.70 | −10.30 | .025773 | .053628 |
| (e) | 0.95 | 0.38 | −0.13 | 0.3 | 7.0 | 9.9 | 5.70 | 5.39 | 8.00 | 6.42 | −2.47 | −1.82 | .175439 | .337211 |
| (f) | 0.52 | 0.51 | 0.16 | 1.4 | 2.9 | 5.0 | 2.96 | 1.53 | 3.96 | 3.31 | −1.17 | −0.47 | .337838 | .654331 |
| (g) | 0.84 | 0.58 | 0.35 | 2.9 | 14.8 | 22.9 | 17.70 | 14.80 | 32.60 | 24.80 | −6.13 | −6.00 | .056497 | .087661 |
| (h) | 0.79 | 0.81 | 0.77 | 0.6 | 1.0 | 1.2 | 2.611 | 2.051 | 4.897 | 5.165 | −0.586 | −0.33 | .382995 | .487567 |
| (i) | 0.79 | 0.43 | 1.00 | 8.5 | 47.0 | 0.1 | 41.20 | 32.70 | 78.80 | 64.10 | 9.10 | −21.70 | .024272 | .037241 |

As Fig. 2 shows, the stationary value $E_4$ exists only within the range:

$$2\alpha_2 < \beta_2 < \alpha_2 + \beta_3/4,$$

and occurs only if $\beta_3/2 > 2\alpha_2$.

Existence of the stationary value $E_5$ is delimited by

$$\alpha_2 < \beta_2 < \min(\beta_3/2, 2\alpha_2),$$

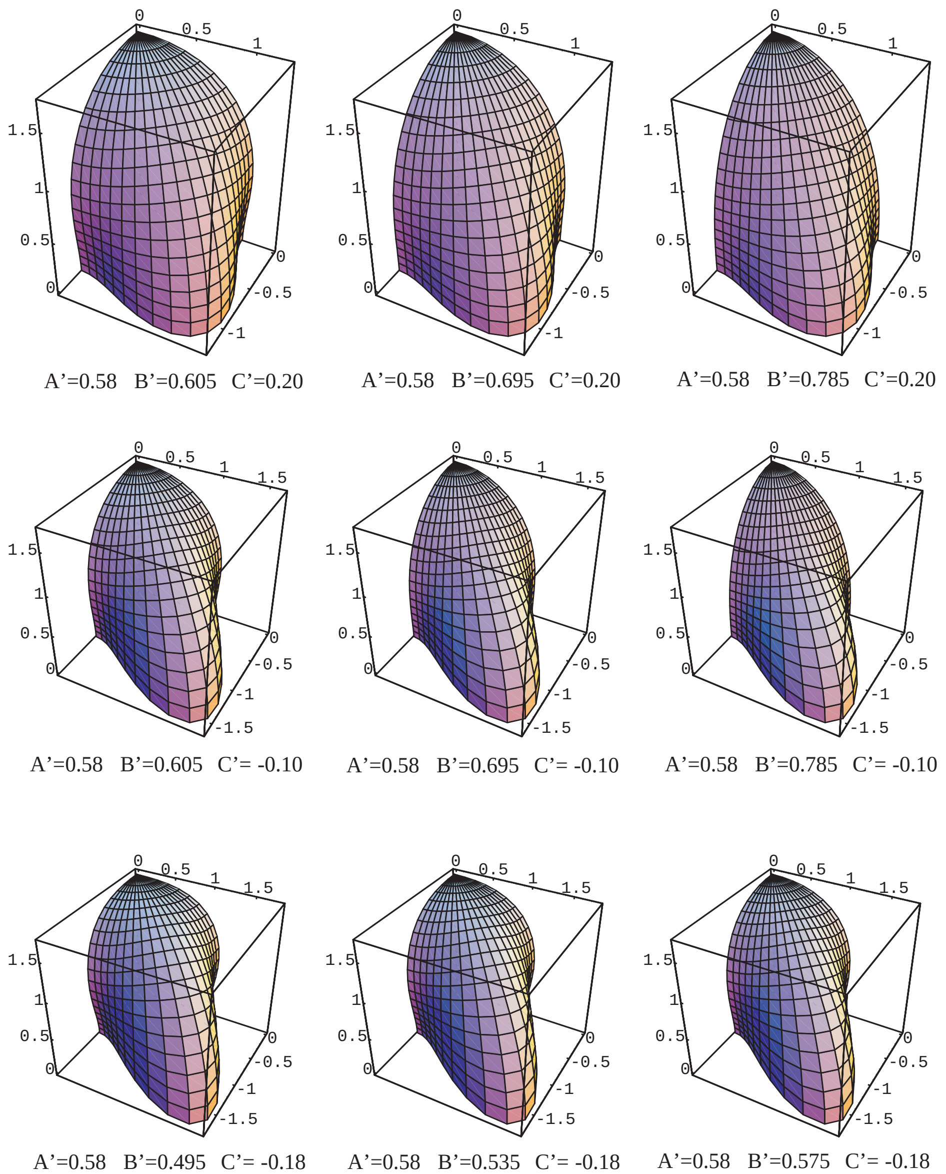


Fig. 5. Class 1b: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

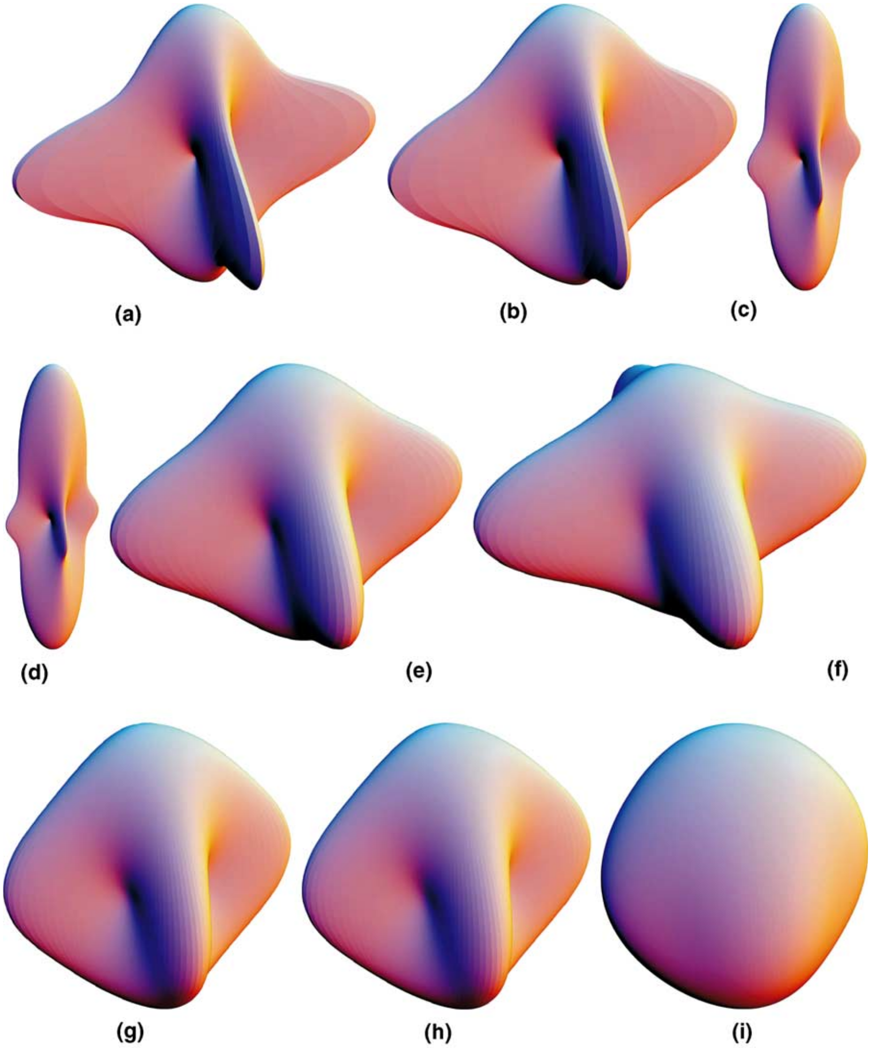

Fig. 6. Tetragonal system: materials belonging to Class 1b, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $TeO_2$ (tellurium dioxide, at 78 K); (b) $TeO_2$ (tellurium dioxide, at 300 K); (c) $Hg_2Br_2$ (mercurous bromide); (d) $Hg_2I_2$ (mercurous iodide); (e) $MnF_2$ (manganese fluoride); (f) α-$NiSO_4 \cdot 6H_2O$ (nickel sulfate hexahydrate); (g) Sn (Tin); (h) $TiO_2$ (titanium dioxide, rutile); (i) $WSi_2$ (tungsten disilicide). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 0.07 | 0.12 | −0.89 | 130.7 | 245.5 | 528.3 | 140.00 | 9.30 | 36.90 | 13.70 | −131.00 | −1.20 | .007143 | .126183 |
| (b) | 0.09 | 0.14 | −0.84 | 106.5 | 201.4 | 431.1 | 117.00 | 10.50 | 37.40 | 15.10 | −106.10 | −2.38 | .008547 | .108401 |
| (c) | 0.03 | 0.13 | −0.80 | 437.8 | 790.4 | 1634.6 | 453.00 | 15.20 | 134.00 | 89.40 | −409.00 | −9.20 | .002208 | .065789 |
| (d) | 0.03 | 0.13 | −0.80 | 525.4 | 938.6 | 1942.5 | 541.00 | 15.60 | 171.00 | 89.50 | −475.00 | −13.80 | .001848 | .064103 |
| (e) | 0.33 | 0.45 | −0.45 | 18.5 | 30.4 | 79.8 | 27.60 | 9.09 | 32.00 | 14.20 | −19.40 | −3.60 | .036232 | .130719 |
| (f) | 0.53 | 0.65 | −0.29 | 30.7 | 46.1 | 167.8 | 65.00 | 34.30 | 86.50 | 56.20 | −47.00 | −1.30 | .015385 | .043384 |
| (g) | 0.35 | 0.44 | −0.27 | 27.6 | 47.8 | 107.5 | 42.40 | 14.80 | 45.60 | 42.10 | −32.40 | −4.30 | .023585 | .067568 |
| (h) | 0.38 | 0.47 | −0.21 | 4.2 | 7.2 | 16.4 | 6.80 | 2.60 | 8.06 | 5.21 | −4.01 | −0.85 | .147059 | .384615 |
| (i) | 0.76 | 0.84 | 0.67 | 0.6 | 0.8 | 1.6 | 2.482 | 1.890 | 4.726 | 4.598 | −0.632 | −0.271 | .402901 | .529101 |

and requires $\min(\beta_3/2, 2\alpha_2) > \alpha_2$. As a consequence, as clearly shown in Fig. 2, it is not possible for this class to have co-existence of *both* stationary values $E_4$ and $E_5$.

Fig. 5 outlines the evolution of the surface $E(\boldsymbol{n})$ when, for a fixed value of $A'$, $B'$ and $C'$ are independently allowed to change.

In Fig. 6 the whole surface of $E(\boldsymbol{n})$ generated by some representative materials belonging to Class 1b is reported.

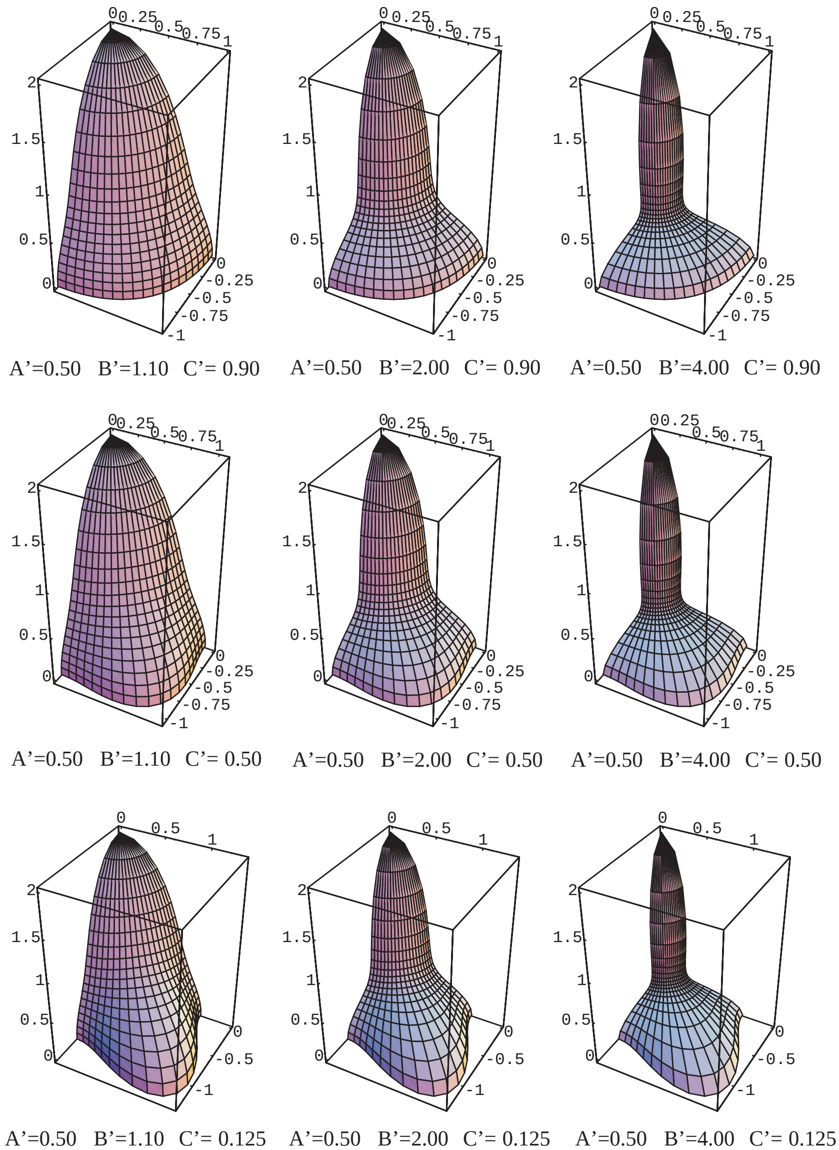


Fig. 7. Class 1c: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

### 3.3. Class 1c

It is delimited as follows:

$$\alpha_2 > 0; \quad \beta_3 > 0; \quad \beta_2 < \alpha_2,$$

or, by switching to dimensionless parameters,

$$A' < 1; \quad C' < 1; \quad B' > \frac{1 + A'}{2}.$$

By direct inspection of Fig. 2, it results that the stationary value $E_4$ is guaranteed to exist if

$$\beta_2 < 0,$$

while $E_5$ exists within the range:

$$\beta_2 < \min(\beta_3/2, 2\alpha_2).$$

Hence *both* stationary values exist if $\beta_2 < 0$.

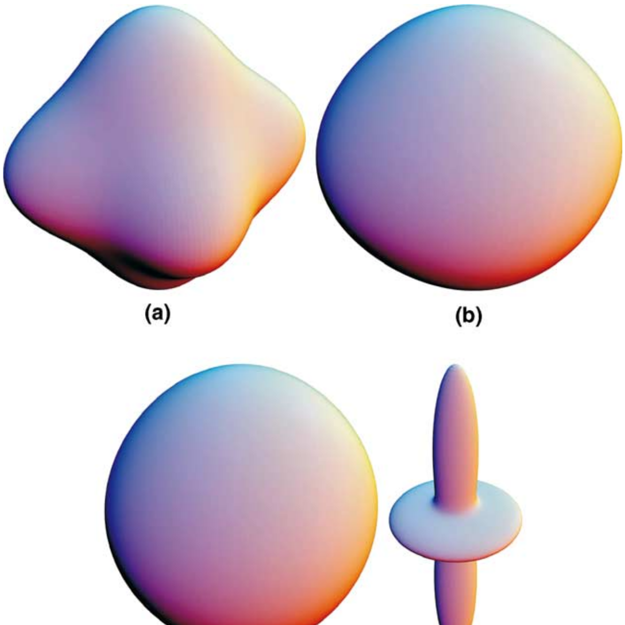


Fig. 8. Tetragonal system: materials belonging to Class 1c, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $Ag_2SO_4 \cdot 4NH_3$ (silver sulfate, ammoniated); (b) indium–lead (atomic percentage Pb: 17%; (c) vesuvian (complex CaMgFeAl silicate); (d) $CsNiF_3$ (cesium nickel fluoride). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 0.72 | 1.12 | 0.50 | 13.0 | −11.2 | 46.0 | 46.40 | 33.40 | 127.00 | 86.00 | −19.60 | −11.50 | .020966 | .029940 |
| (b) | 1.00 | 1.11 | 0.93 | 0.0 | −11.0 | 7.0 | 49.00 | 49.00 | 153.00 | 123.00 | −16.00 | −22.00 | .019324 | .021164 |
| (c) | 0.90 | 0.99 | 0.97 | 0.8 | 0.2 | 0.5 | 7.55 | 6.80 | 17.90 | 18.50 | −1.93 | −1.49 | .132450 | .147059 |
| (d) | 0.38 | 3.60 | 0.99 | 18.1 | −151.2 | 0.4 | 29.10 | 11.00 | 213.00 | 84.00 | −13.10 | −1.81 | .015910 | .090909 |

Fig. 7 shows the evolution of the surface $E(\boldsymbol{n})$ when $B'$ and $C'$ do vary independently of each other, while $A'$ is kept fixed.

The whole surface representing $E(\boldsymbol{n})$ for four materials belonging to Class 1$c$ is depicted in Fig. 8.

### 3.4. Class 1d

The material parameters ranges are, for this class, as shown in Table 4:

$$\alpha_2 > 0; \quad \beta_3 < 0; \quad \beta_2 > \alpha_2,$$

whereas the dimensionless counterparts of $\alpha_2$, $\beta_2$, $\beta_3$ provide:

$$A' < 1; \quad C' > 1; \quad B' < \frac{1 + A'}{2}.$$

In Fig. 9, along with the extension of contiguous classes 1d, 1e and 1f as a function of the material parameter $\beta_2$, it is shown that the stationary value $E_4$ exists when

$$2\alpha_2 < \beta_2 < \beta_2^\star,$$

whereas the existence of $E_5$ requires that

$$2\alpha_2 < \beta_2 < \beta_2^{\star\star}.$$

For this given class, it is easy to realize that $\beta_2^\star < \beta_2^{\star\star}$, so that stationary values $E_4$ and $E_5$ coexist if

$$2\alpha_2 < \beta_2 < \beta_2^\star.$$

The evolution of the surface representing $E(\boldsymbol{n})$ is shown in Fig. 10 when $A'$ is kept fixed, while $B'$ and $C'$ are allowed to change.

The whole surface of $E(\boldsymbol{n})$ for one representative material belonging to Class 1d is shown in Fig. 11.

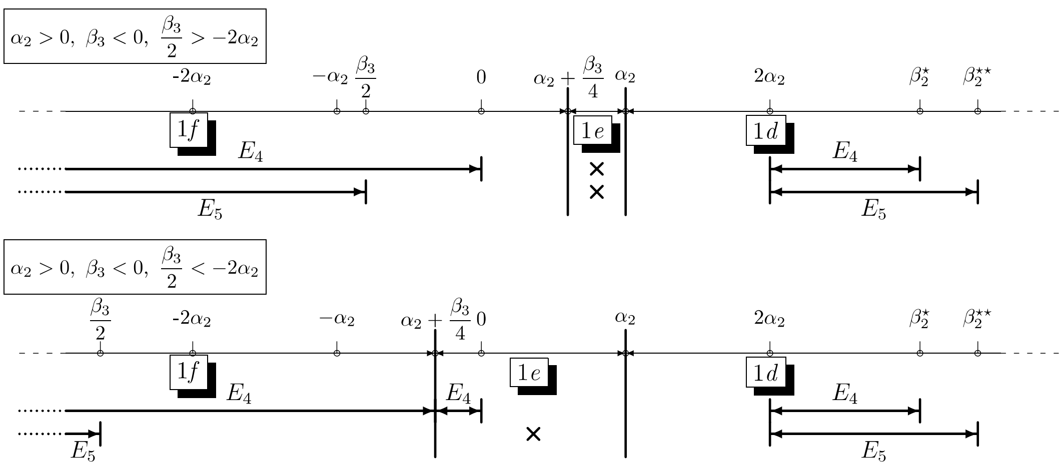


Fig. 9. Extension of Classes 1d, 1e, 1f as a function of the material parameter $\beta_2$ and corresponding ranges of existence of stationary values $E_4$ and $E_5$.

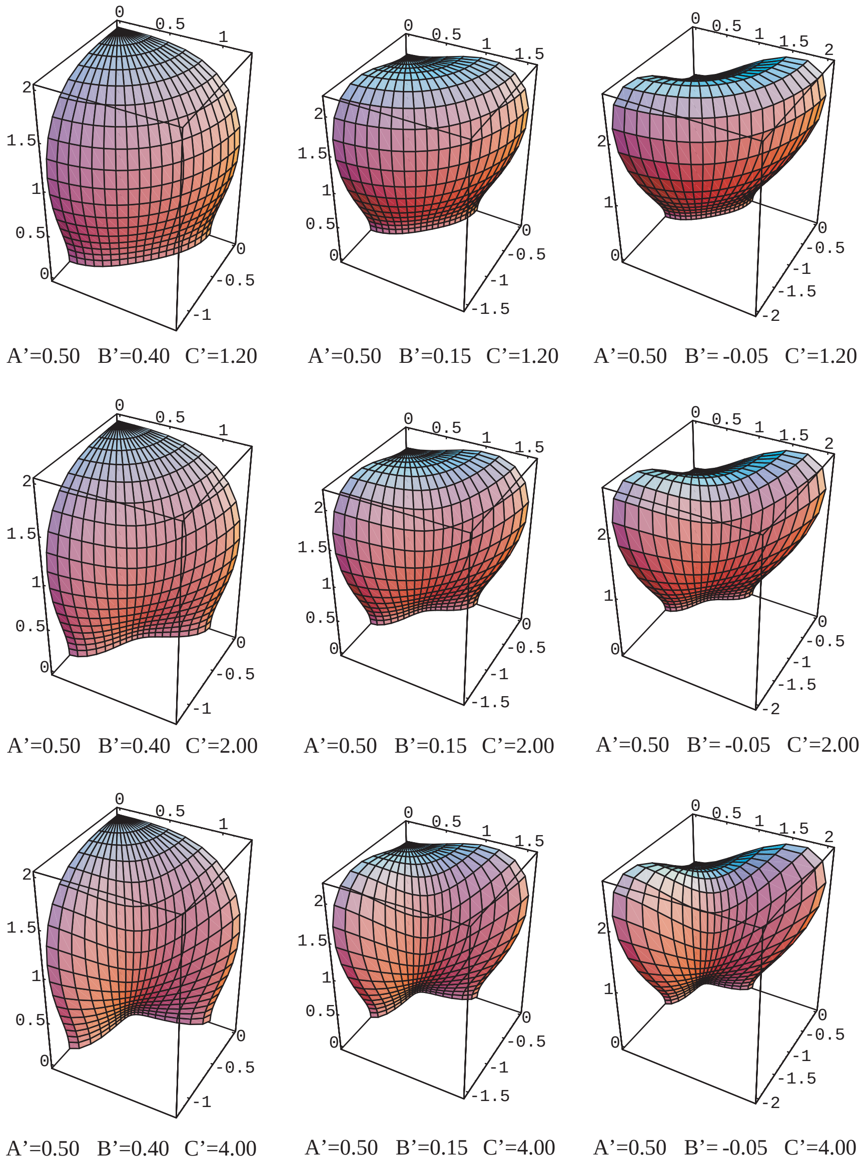


Fig. 10. Class 1d: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

### 3.5. Class 1e

This class is delimited as follows:

$$\alpha_2 > 0; \quad \beta_3 < 0; \quad \alpha_2 + \beta_3/4 < \beta_2 < \alpha_2,$$

or, by using dimensionless parameters,

$$A' < 1; \quad C' > 1; \quad \frac{1 + A'}{2} < B' < \frac{1 + C' + 2A'}{4}.$$

Fig. 11. Tetragonal system: material belonging to Class 1d, defined in Table 4. $(NH_2)_2CO$ (urea): $A' = 0.67$, $B' = 0.32$ and $C' = 11.86$ (dimensionless); $\alpha_2 = 31.0$, $\beta_2 = 130.0$, $\beta_3 = -2062.0$ (expressed in $TPa^{-1}$). Elastic compliance coefficients (taken from Landolt and Börnstein, 1992 and expressed in $TPa^{-1}$): $s_{11} = 95.10$, $s_{33} = 64.00$, $s_{44} = 160.00$, $s_{66} = 2220.00$, $s_{12} = 16.00$, $s_{13} = -50.00$. Young's moduli (in GPa): $E_{\min} = .001638$ and $E_{\max} = .019112$.

Fig. 9 ensures the existence of the stationary value $E_4$ when

$$\alpha_2 + \beta_3/4 < \beta_2 < 0,$$

but this happens only if $\alpha_2 + \beta_3/4 < 0$.

The stationary value $E_5$ does not occur in any case within this class; as a consequence that prevents coexistence of *both* $E_4$ and $E_5$.

The evolution of the surface $E(\boldsymbol{n})$ when parameters $B'$ and $C'$ are allowed to change independently of each other is shown in Fig. 12.

Fig. 13 provides the complete surface generated by the directional dependence of Young's modulus for four different materials belonging to this class.

### 3.6. Class 1f

It is delimited as follows:

$$\alpha_2 > 0; \quad \beta_3 < 0; \quad \beta_2 < \alpha_2 + \beta_3/4,$$

or, in dimensionless form,

$$A' < 1; \quad C' > 1; \quad B' > \frac{1 + C' + 2A'}{4}.$$

By referring to Fig. 9, the stationary value $E_4$ exists when

$$\beta_2 < \min(0, \alpha_2 + \beta_3/4),$$

whereas the other one, $E_5$, exists if

$$\beta_2 < \beta_3/2.$$

*Coexistence* of these stationary values occurs therefore in the range $\beta_2 < \beta_3/2$.

Fig. 14 shows how the surface representing $E(\boldsymbol{n})$ changes when, for a fixed value of $A'$, $B'$ and $C'$ do change independently.

In Fig. 15, Young's modulus generated surfaces for some materials belonging to the considered class are shown.

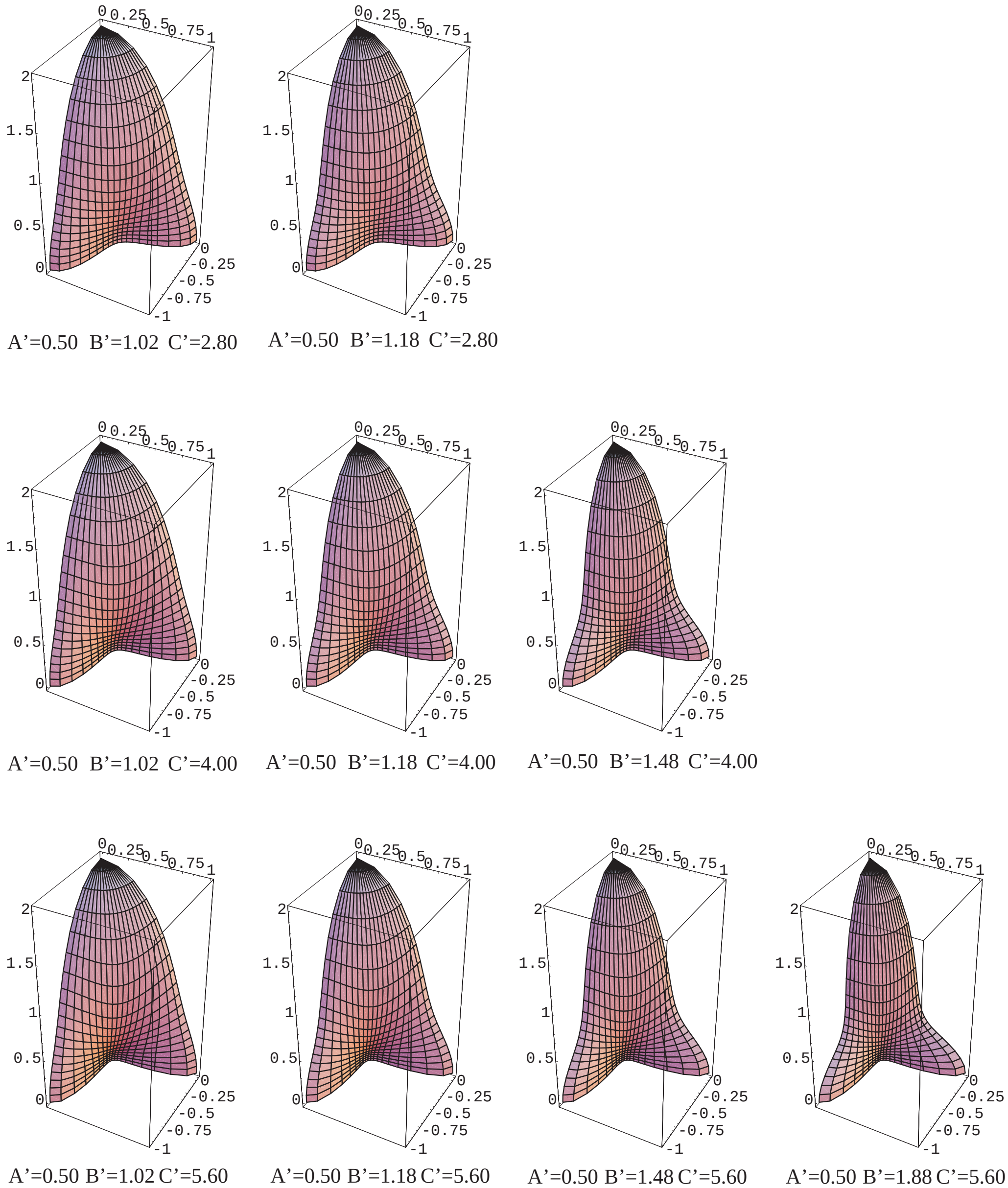


Fig. 12. Class 1e: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

### 3.7. Class 2a

It is delimited as follows (see Table 4):

$$\alpha_2 < 0; \quad \beta_3 > 0; \quad \beta_2 > \alpha_2 + \beta_3/4,$$

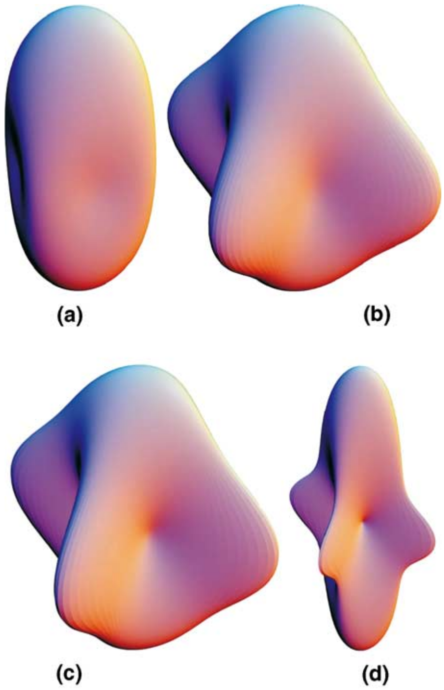


Fig. 13. Tetragonal system: materials belonging to Class 1e, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $Zr_2Ni$ (zirconium nickel); (b) $ZrSiO_4$ (zircon); (c) $LuPO_4$ (lutetium phosphate); (d) $(NH_2)_2CO$ (urea). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 0.50 | 0.82 | 1.67 | 10.8 | 7.9 | −28.8 | 21.60 | 10.80 | 41.70 | 104.00 | −16.00 | −3.20 | .034722 | .092593 |
| (b) | 0.94 | 1.39 | 3.84 | 0.2 | −2.1 | −15.0 | 2.65 | 2.50 | 8.85 | 20.70 | −0.18 | −0.75 | .156006 | .400000 |
| (c) | 0.93 | 1.39 | 6.58 | 0.3 | −2.7 | −39.1 | 3.50 | 3.25 | 11.80 | 46.10 | −0.02 | −1.05 | .075386 | .307692 |
| (d) | 0.48 | 1.62 | 22.34 | 23.2 | −56.0 | −1916.6 | 44.90 | 21.70 | 160.00 | 2000.00 | 3.20 | −7.10 | .001908 | .046083 |

or, by switching to dimensionless form,

$$A' > 1; \quad C' < 1; \quad B' < \frac{1 + C' + 2A'}{4}.$$

As shown in Fig. 16, where the extension of the three contiguous classes 2a, 2b and 2c are marked as a function of the material parameter $\beta_2$, existence of stationary value $E_4$ is ensured within the range:

$$\max(\alpha_2 + \beta_3/4, 0) < \beta_2 < \beta_2^{\star}$$

and that of stationary value $E_5$ within the range:

$$\beta_3/2 < \beta_2 < \beta_2^{\star\star}.$$

As it is easy to check, for the present class it results $\beta_2^{\star\star} < \beta_2^{\star}$; hence the *simultaneous* occurrence of $E_4$ and $E_5$ is guaranteed if

$$\beta_3/2 < \beta_2 < \beta_2^{\star\star}.$$

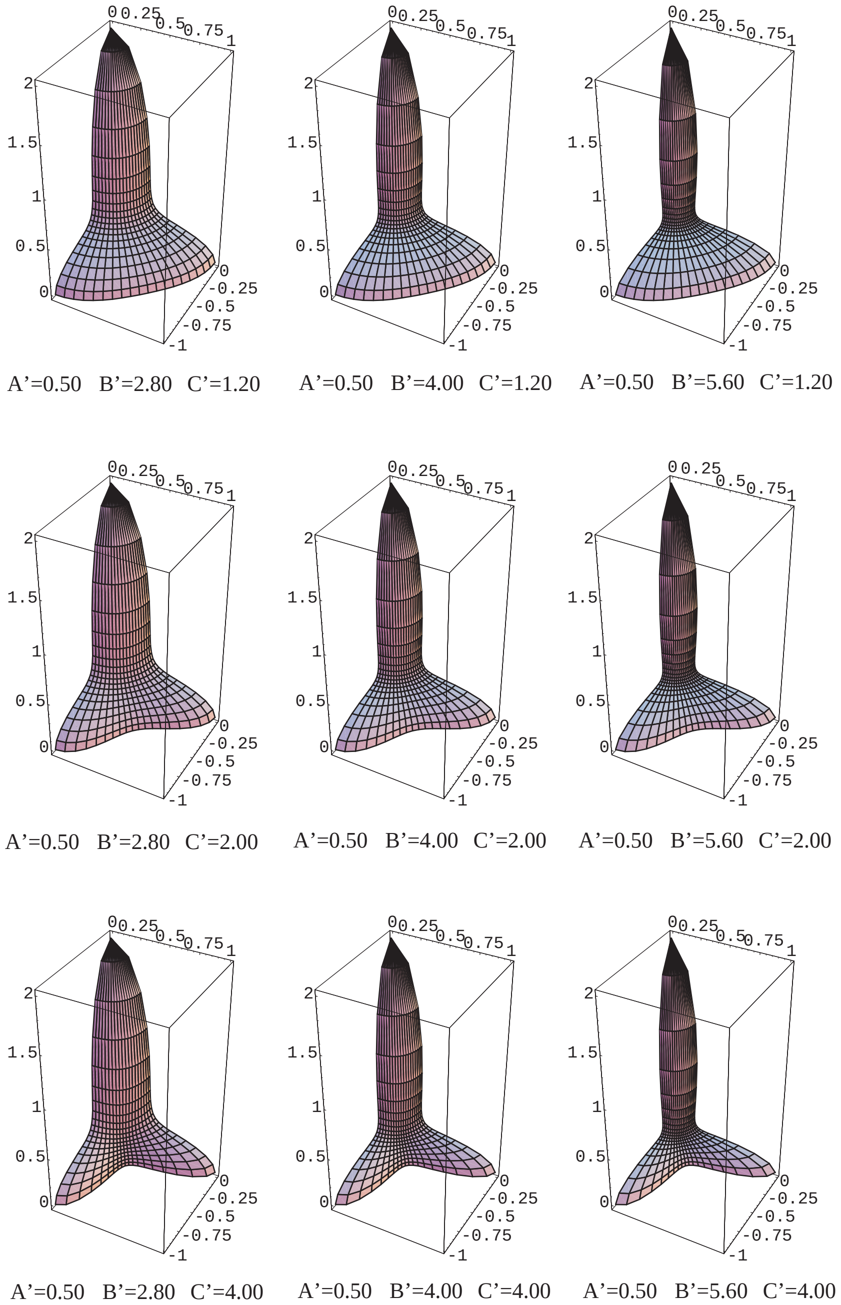


Fig. 14. Class 1f: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

The evolution of surface $E(\boldsymbol{n})$ when parameters $B'$ and $C'$ are allowed to change independently, while $A'$ is fixed, is depicted in Fig. 17.

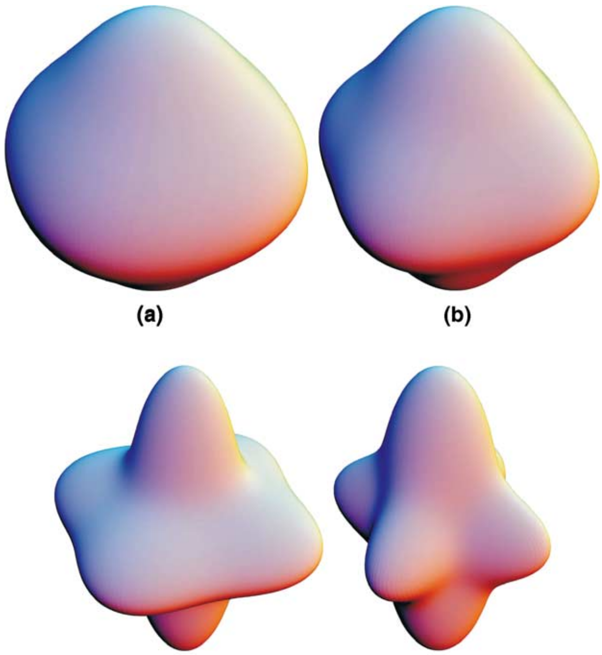


Fig. 15. Tetragonal system: materials belonging to Class 1f, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $Sr_{0.45}Ba_{0.55}Nb_2O_6$ (strontium barium niobate); (b) $K_2CuF_4$ (potassium copper fluoride); (c) scapolite (complex aluminosilicate); (d) $LiRb_5(SO_4)_3 \cdot 1\frac{1}{2}H_2SO_4$ (lithium rubidium sulfate trihydrogen sulfate). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 0.98 | 1.34 | 1.13 | 0.1 | −4.4 | −1.7 | 6.44 | 6.34 | 19.23 | 19.23 | −2.33 | −0.97 | .130893 | .157729 |
| (b) | 0.92 | 1.42 | 1.32 | 1.4 | −14.7 | −11.4 | 17.60 | 16.20 | 62.50 | 45.40 | 0.60 | −6.30 | .045499 | .061728 |
| (c) | 0.85 | 2.37 | 1.50 | 1.8 | −33.7 | −12.4 | 12.30 | 10.50 | 63.90 | 43.70 | −3.37 | −2.79 | .047088 | .095238 |
| (d) | 0.71 | 1.78 | 2.48 | 10.7 | −58.0 | −110.8 | 37.40 | 26.70 | 143.00 | 222.00 | −18.20 | −5.10 | .015351 | .037453 |

Fig. 18 presents the whole surface generated by the directional dependence of Young's modulus for 9 representative materials belonging to Class 2a.

### 3.8. Class 2b

In this case, the range of material parameters is as follows:

$$\alpha_2 < 0; \quad \beta_3 > 0; \quad \alpha_2 < \beta_2 < \alpha_2 + \beta_3/4,$$

or, by making use of the dimensionless counterparts,

$$A' > 1; \quad C' < 1; \quad \frac{1 + C' + 2A'}{4} < B' < \frac{1 + A'}{2}.$$

By inspection of Fig. 16 it is easily seen that existence of stationary value $E_4$ is ensured within the range:

$$0 < \beta_2 < \alpha_2 + \beta_3/4,$$

which is admissible only if $\beta_3/2 > 0$.

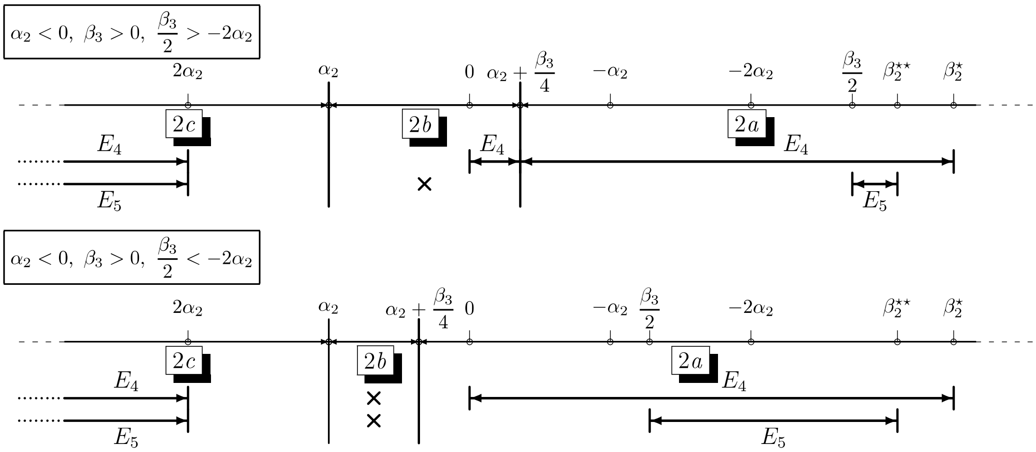


Fig. 16. Extension of Classes 2a, 2b, 2c as a function of the material parameter $\beta_2$ and corresponding ranges of existence of stationary values $E_4$ and $E_5$.

For this class the other stationary value, $E_5$, does not exist, nor it is possible to find a common range of existence for both values.

Fig. 19 presents, as usual, some parametric plots of the surface $E(\boldsymbol{n})$ to allow understanding the effects of changes of parameters $B'$ and $C'$ (whereas $A'$ is kept fixed), on its shape evolution. These shapes should be compared with the whole surface produced by a material belonging to the present class, shown in Fig. 20.

### 3.9. Class 2c

These are the bounds delimiting this class:

$$\alpha_2 < 0; \quad \beta_3 > 0; \quad \beta_2 < \alpha_2,$$

and their dimensionless counterparts turn out to be

$$A' > 1; \quad C' < 1; \quad B' > \frac{1 + A'}{2}.$$

Fig. 16 shows that the stationary values $E_4$ and $E_5$ *simultaneously* exist if

$$\beta_2 < 2\alpha_2.$$

Parametric plots showing the evolution of surface $E(\boldsymbol{n})$ when parameters $B'$ and $C'$ are allowed to change by taking $A'$ fixed are presented in Fig. 21.

Fig. 22 provides instead the complete surfaces produced by the directional dependence of Young's modulus for some materials belonging to Class 2c.

### 3.10. Class 2d

The range of material parameters for the present class are

$$\alpha_2 < 0; \quad \beta_3 < 0; \quad \beta_2 > \alpha_2,$$

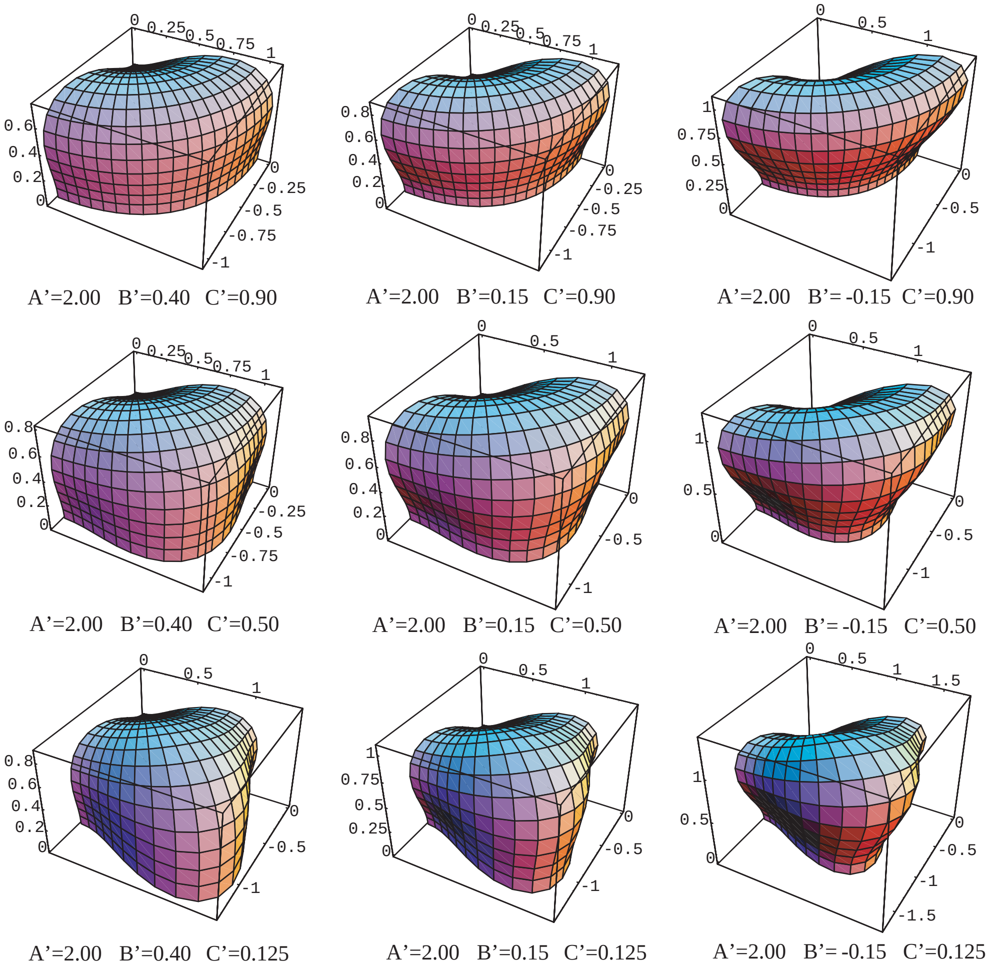


Fig. 17. Class 2*a*: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

or, alternatively,

$$A' > 1; \quad C' > 1; \quad B' < \frac{1 + A'}{2}.$$

Fig. 23 outlines the extension of the contiguous classes 2d, 2e and 2f as a function of the material parameter $\beta_2$; it provides moreover the range of existence of the stationary value $E_4$, which turns out to be

$$0 < \beta_2 < \beta_2^{\star}$$

and that of the stationary value $E_5$, namely

$$\max(\alpha_2, \beta_3/2) < \beta_2 < \beta_2^{\star\star}.$$

It is easy to check that in this case $\beta_2^{\star} < \beta_2^{\star\star}$; hence $E_4$ and $E_5$ *coexist* if

$$0 < \beta_2 < \beta_2^{\star}.$$

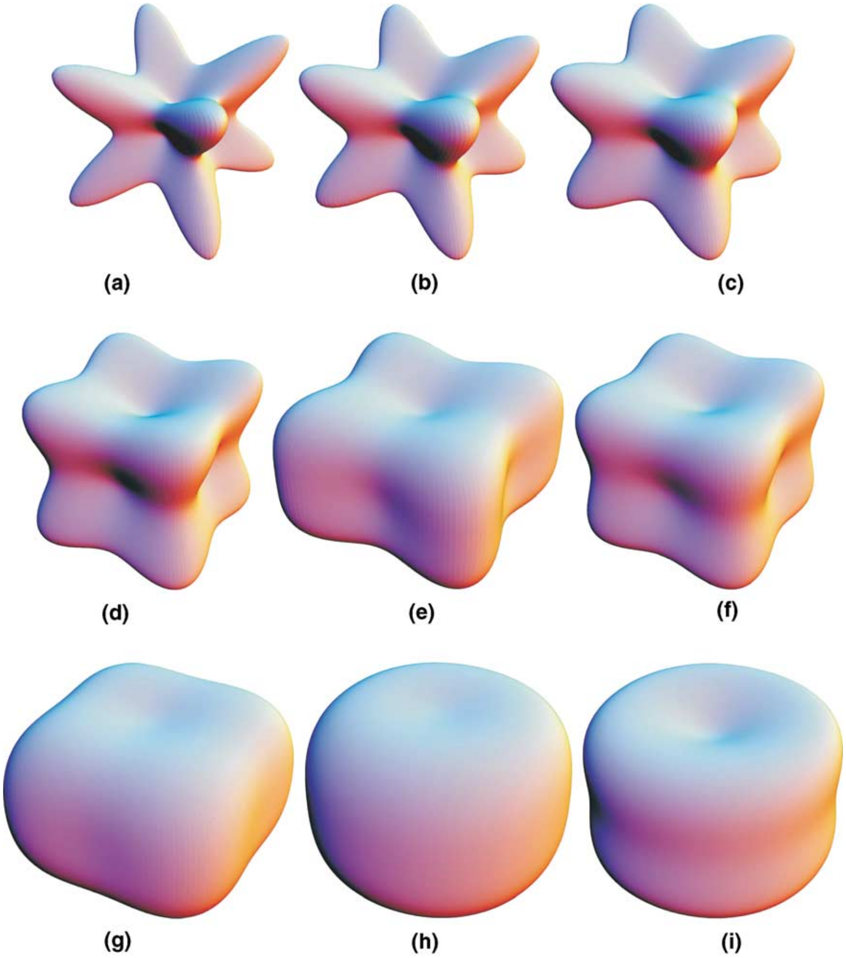


Fig. 18. Tetragonal system: materials belonging to Class 2a, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) In–Tl (indium–thallium, atomic percentage Tl: 10%); (b) In–Tl (indium–thallium, atomic percentage Tl: 11.5%); (c) In (indium); (d) $CdGeAs_2$ (cadmium germanium arsenide); (e) $BaTi_2$ (barium titanate); (f) $AgGaS_2$ (silver gallium sulfide); (g) $BaLaGa_3O_7$ (barium lanthanum gallate); (h) SrClF (strontium chloride fluoride); (i) BaClF (barium chloride fluoride). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 1.07 | −0.25 | −0.22 | −15.0 | 559.0 | 547.0 | 224.00 | 239.00 | 125.00 | 93.00 | −96.00 | −118.00 | .004184 | .024603 |
| (b) | 1.17 | −0.17 | −0.15 | −32.0 | 441.0 | 431.0 | 188.00 | 220.00 | 147.00 | 95.00 | −75.00 | −106.00 | .004545 | .022005 |
| (c) | 1.32 | −0.12 | −0.03 | −47.4 | 332.9 | 306.4 | 148.80 | 196.20 | 153.70 | 83.20 | −46.00 | −94.50 | .005097 | .021921 |
| (d) | 1.25 | 0.07 | 0.24 | −5.3 | 40.2 | 32.8 | 21.60 | 26.90 | 23.80 | 24.50 | −7.04 | −10.40 | .037175 | .104104 |
| (e) | 1.95 | 0.49 | 0.26 | −7.7 | 8.2 | 12.0 | 8.05 | 15.70 | 18.40 | 8.84 | −2.35 | −5.24 | .063694 | .201379 |
| (f) | 1.37 | 0.24 | 0.33 | −9.7 | 39.9 | 35.3 | 26.20 | 35.90 | 41.50 | 32.50 | −7.70 | −14.50 | .027855 | .069737 |
| (g) | 1.61 | 0.77 | 0.59 | −6.1 | 4.5 | 8.1 | 10.03 | 16.16 | 25.64 | 18.52 | −3.30 | −5.06 | .061881 | .125178 |
| (h) | 1.51 | 0.81 | 0.96 | −6.4 | 4.7 | 1.0 | 12.50 | 18.90 | 32.70 | 26.40 | −1.20 | −6.20 | .052910 | .084434 |
| (i) | 2.02 | 0.55 | 0.97 | −16.5 | 14.5 | 1.1 | 16.20 | 32.70 | 41.10 | 30.10 | 0.60 | −11.60 | .030581 | .069727 |

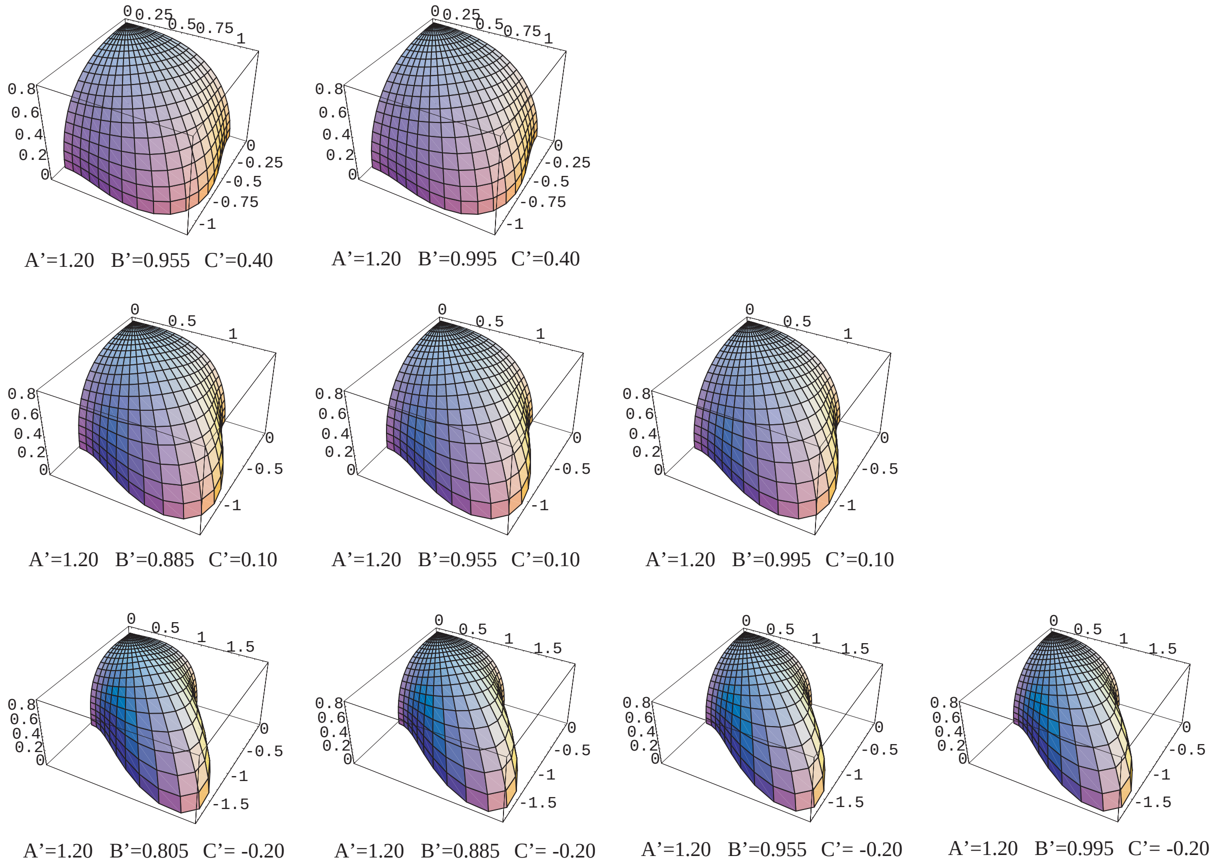


Fig. 19. Class 2b: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

Fig. 20. Tetragonal system: material belonging to Class 2b, defined in Table 4. $Pb_{0.37}Ba_{0.63}Nb_2O_6$ (lead barium niobate): $A' = 1.66$, $B' = 1.28$ and $C' = 0.78$ (dimensionless); $\alpha_2 = -3.8$, $\beta_2 = -3.2$, $\beta_3 = 2.6$ (expressed in $TPa^{-1}$); Elastic compliance coefficients (taken from Landolt and Börnstein, 1992 and expressed in $TPa^{-1}$): $s_{11} = 5.80$, $s_{33} = 9.60$, $s_{44} = 18.20$, $s_{66} = 12.40$, $s_{12} = -1.70$, $s_{13} = -1.70$. Young's moduli (in GPa): $E_{min} = .104167$ and $E_{max} = .194175$.

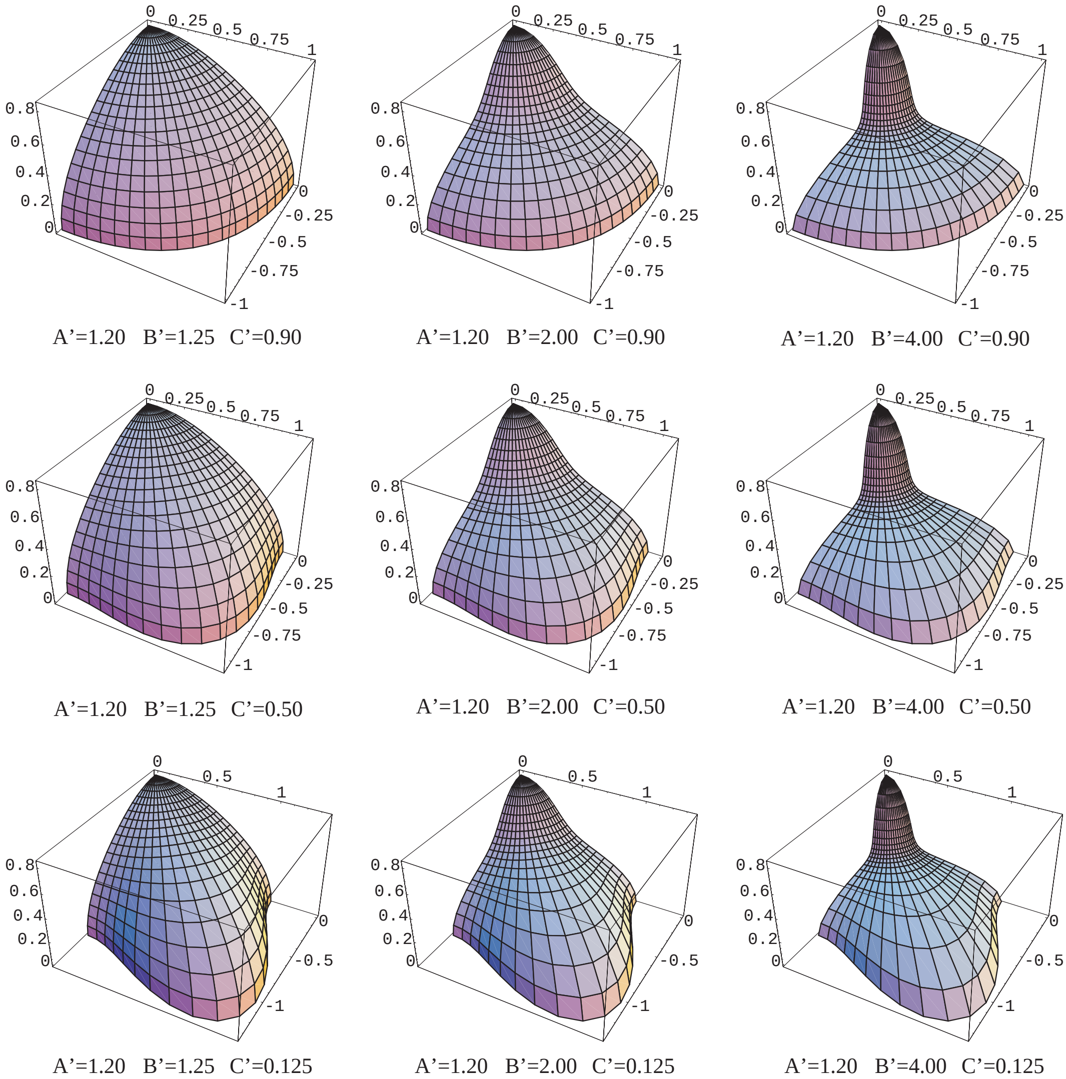


Fig. 21. Class 2c: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

Fig. 24 shows parametric plots of the surface $E(\boldsymbol{n})$, and is intended to provide some clues about the evolution of the surface shape when, for a fixed value of $A'$, the dimensionless parameters $B'$ and $C'$ are allowed to change.

Some complete surfaces generated by Young's modulus for several materials belonging to this class are presented in Fig. 25.

### 3.11. Class 2e

The delimiting range for material parameters is

$$\alpha_2 < 0; \quad \beta_3 < 0; \quad \alpha_2 + \beta_3/4 < \beta_2 < \alpha_2,$$

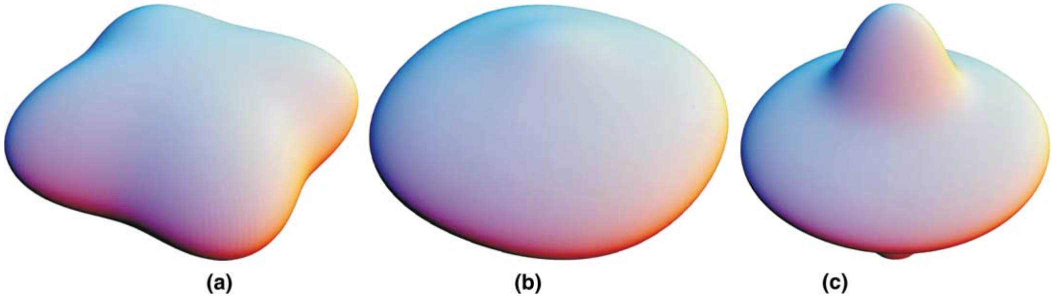


Fig. 22. Tetragonal system: materials belonging to Class 2c, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $Pb_{0.346}Ba_{0.590}Na_{0.036}Li_{0.028}$–$Nb_2O_6$ (lead barium niobate, Na, Li-doped); (b) $Ba_2Si_2TiO_8$ (barium silicon titanium oxide, fresnoite); (c) $PdPb_2$ (palladium plumbide). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 1.57 | 1.34 | 0.51 | −3.0 | −3.5 | 5.0 | 5.10 | 8.10 | 13.40 | 8.80 | −1.80 | 0.14 | .123457 | .259740 |
| (b) | 1.08 | 1.75 | 0.91 | −5.4 | −11.4 | 1.4 | 7.60 | 13.00 | 30.00 | 17.00 | −1.60 | −1.70 | .076834 | .137931 |
| (c) | 1.02 | 2.18 | 0.98 | −0.0 | −3.5 | 0.1 | 1.48 | 1.50 | 7.45 | 4.01 | −0.56 | −0.51 | .425164 | .684229 |

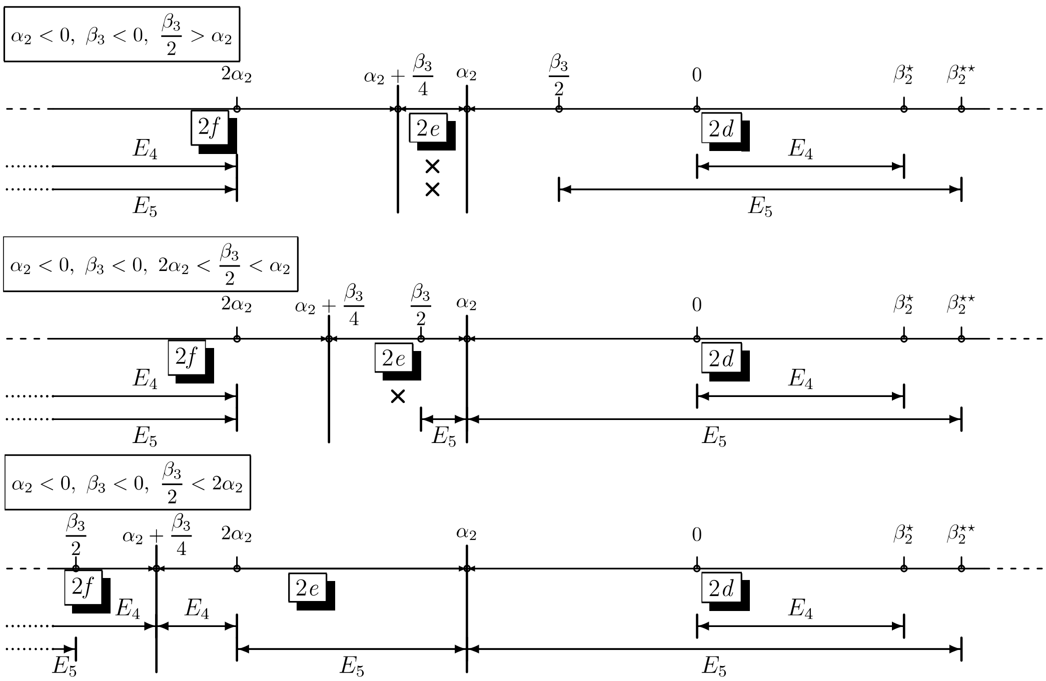


Fig. 23. Extension of Classes 2d, 2e, 2f as a function of the material parameter $\beta_2$ and corresponding ranges of existence of stationary values $E_4$ and $E_5$.

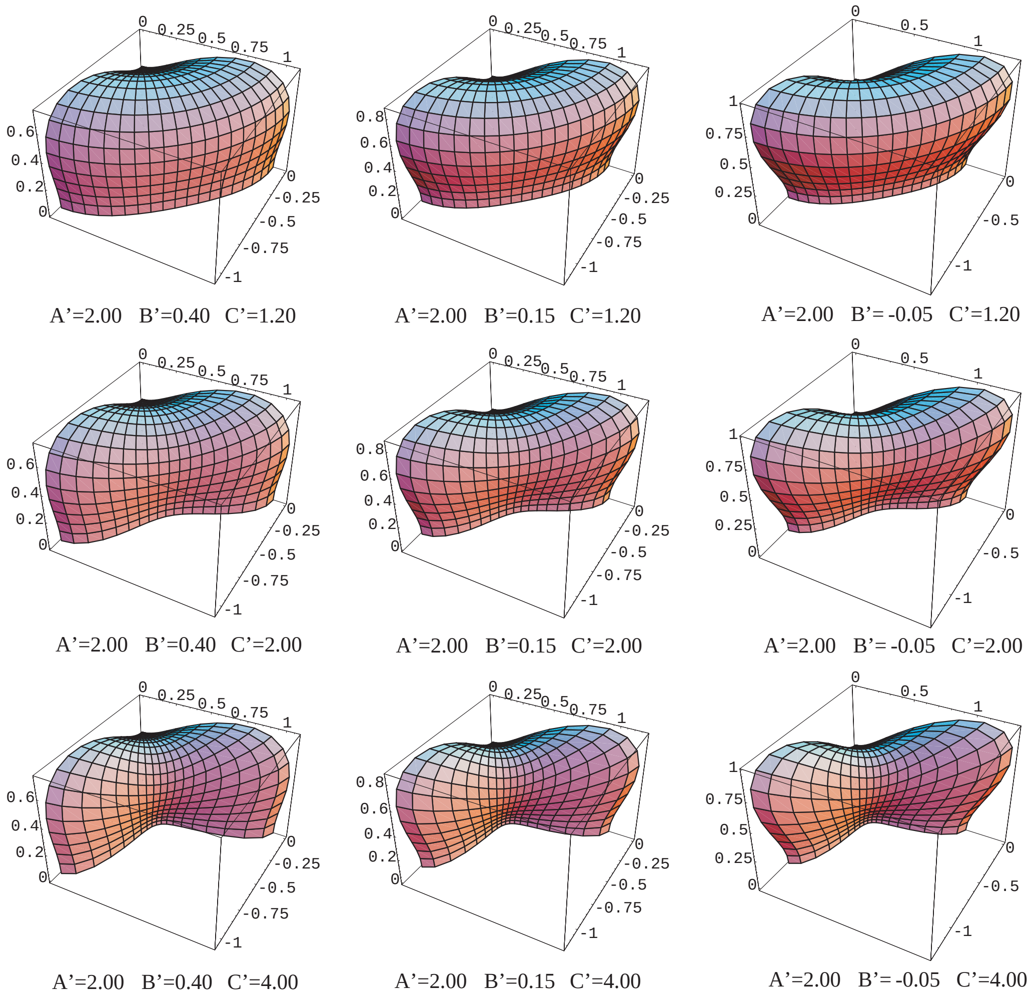


Fig. 24. Class 2d: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

and for the dimensionless ones is instead:

$$A' > 1; \quad C' > 1; \quad \frac{1 + A'}{2} < B' < \frac{1 + C' + 2A'}{4}.$$

By inspection of Fig. 23 it turns out that the stationary value $E_4$ exists if

$$\alpha_2 + \beta_3/4 < \beta_2 < 2\alpha_2,$$

but this requires $\alpha_2 + \beta_3/4 < 2\alpha_2$.

The other stationary value, $E_5$, exists instead within the range

$$\max(2\alpha_2, \beta_3/2) < \beta_2 < \alpha_2,$$

provided that $\beta_3/2 > \alpha_2$.

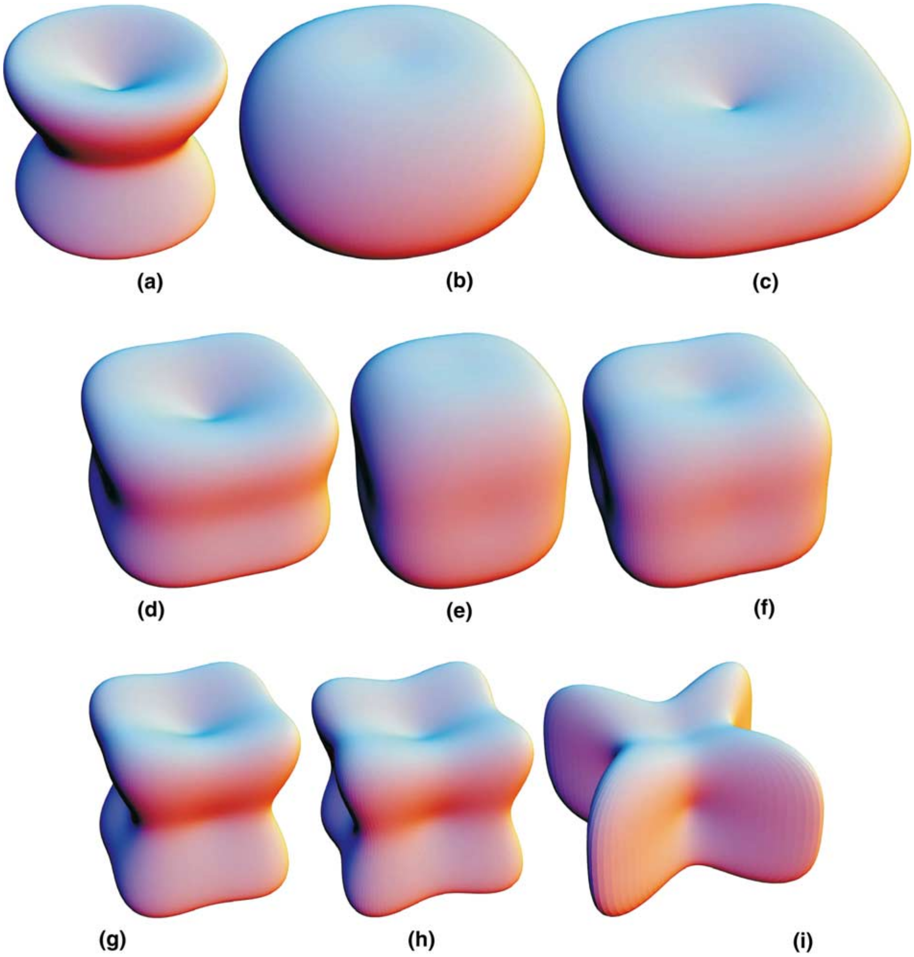


Fig. 25. Tetragonal system: materials belonging to Class 2d, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $Tb_2(Mo)_4)_3$ (terbium molybdate, at 533 K); (b) $Sr_{0.61}Ba_{0.39}Nb_2O_6$ (strontium barium niobate); (c) $FeGe_2$ (iron germanide); (d) $Li_2B_4O_7$ (lithium tetraborate); (e) $Ca_2Sr(C_2H_5CO_2)_6$ (calcium strontium propionate); (f) $C(CH_2)ONO_2)_4$ (pentaerythritol tetranitrate); (g) TlSe (thallium selenide); (h) $Zn[C(NH_2)_3]_2(SO_4)_2$ (Zinc guanidinium sulfate); (i) $HgI_2$ (mercuric iodide). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 1.59 | −0.10 | 1.05 | −14.1 | 53.1 | −2.6 | 24.10 | 32.80 | 37.80 | 34.80 | 8.00 | −21.30 | .026178 | .073473 |
| (b) | 1.90 | 1.13 | 1.08 | −4.8 | −1.4 | −0.8 | 5.32 | 10.10 | 15.50 | 14.40 | −1.46 | −1.73 | .099010 | .187970 |
| (c) | 11.30 | 1.41 | 1.23 | −49.2 | −3.9 | −2.2 | 4.78 | 54.00 | 17.20 | 11.40 | 0.18 | −1.86 | .018519 | .212984 |
| (d) | 2.70 | 0.38 | 1.34 | −15.1 | 11.1 | −6.1 | 8.90 | 24.00 | 17.50 | 21.50 | 1.20 | −5.40 | .041667 | .129461 |
| (e) | 1.04 | 0.70 | 1.38 | −5.0 | 84.0 | −107.0 | 142.00 | 147.00 | 288.00 | 513.00 | −61.00 | −44.00 | .005926 | .008185 |
| (f) | 1.74 | 0.67 | 1.53 | −59.0 | 53.0 | −84.0 | 80.00 | 139.00 | 199.00 | 254.00 | −5.00 | −46.00 | .007194 | .013563 |
| (g) | 1.26 | 0.14 | 1.67 | −6.9 | 45.4 | −35.7 | 26.50 | 33.40 | 31.20 | 83.30 | 2.70 | −11.80 | .028229 | .060069 |
| (h) | 1.60 | 0.22 | 2.14 | −27.4 | 71.4 | −104.1 | 45.56 | 72.92 | 81.17 | 182.82 | 6.21 | −30.70 | .013714 | .030612 |
| (i) | 2.63 | 0.88 | 5.39 | −67.0 | 10.0 | −360.2 | 41.00 | 108.00 | 138.00 | 433.00 | 4.60 | −33.00 | .007631 | .024585 |

As a consequence, such ranges are separate, therefore no *simultaneous* occurrence of $E_4$ and $E_5$ can be foreseen.

Fig. 26 shows the evolution of the surface $E(\boldsymbol{n})$ when, for a fixed value of $A'$, dimensionless parameters $B'$ and $C'$ are independently changed.

Fig. 27 shows instead the surface generated by the directional dependence of Young's modulus for 4 materials belonging to Class 2e.

### 3.12. Class 2f

This last class is delimited as follows (see Table 4):

$$\alpha_2 < 0; \quad \beta_3 < 0; \quad \beta_2 < \alpha_2 + \beta_3/4,$$

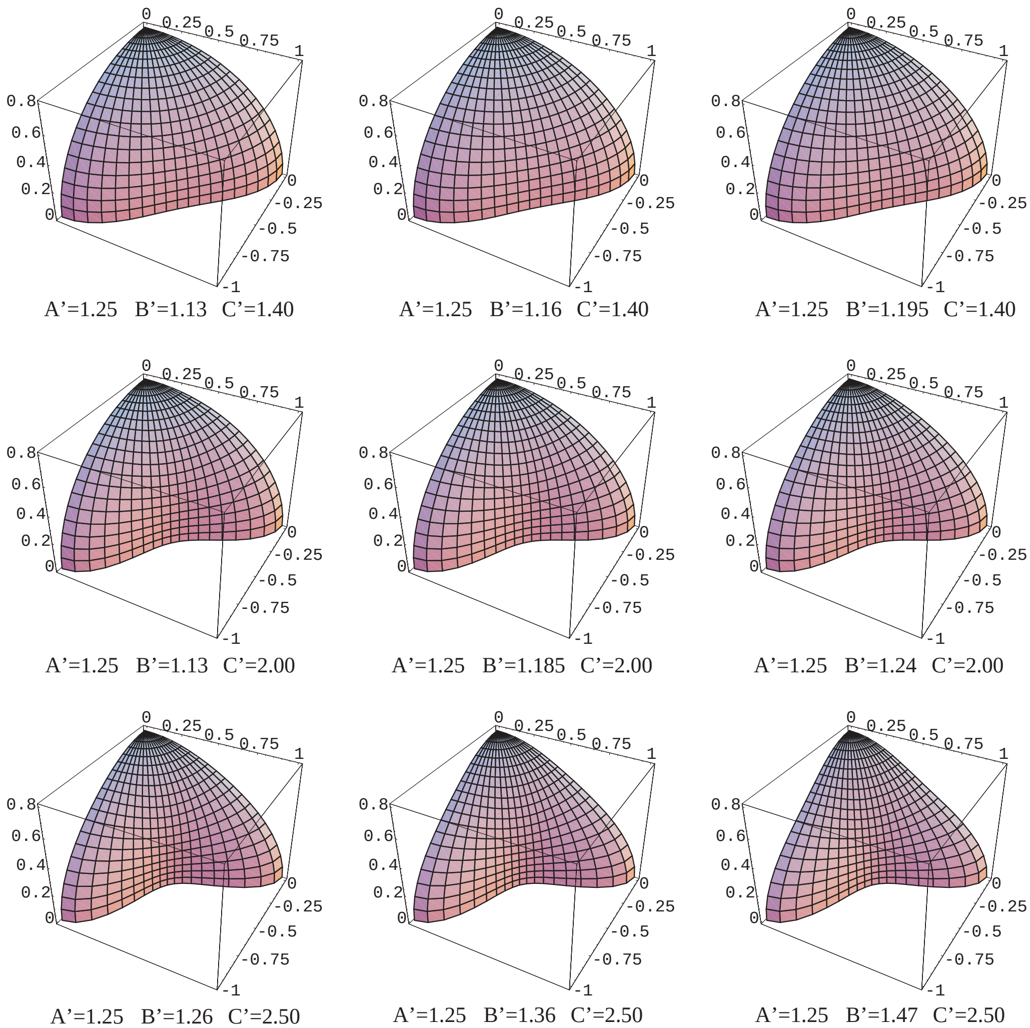


Fig. 26. Class 2e: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

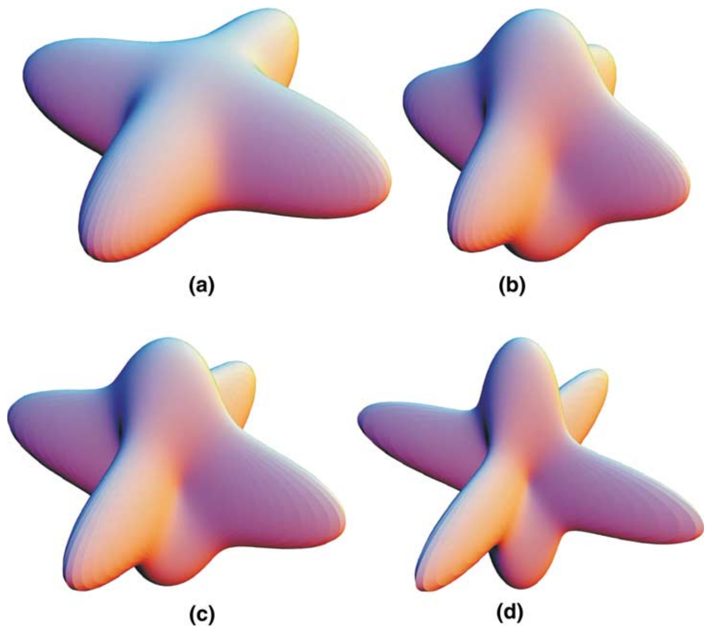


Fig. 27. Tetragonal system: materials belonging to Class 2e, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $ND_4D_2PO_4$ (ammonium dihydrogen phosphate, deuterated); (b) $RbD_2AsO_4$ (rubidium dideuterium arsenate); (c) $RbH_2PO_4$ (rubidium dihydrogen phosphate); (d) $CsH_2AsO_4$ (cesium dihydrogen arsenate). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 2.32 | 2.32 | 4.40 | −25.0 | −50.0 | −129.0 | 19.00 | 44.00 | 110.00 | 163.00 | 2.00 | −11.00 | .019512 | .052632 |
| (b) | 1.10 | 1.97 | 5.39 | −2.4 | −47.8 | −216.8 | 24.70 | 27.10 | 106.00 | 246.00 | 10.10 | −4.40 | .012674 | .040486 |
| (c) | 1.31 | 2.56 | 8.46 | −5.2 | −52.7 | −252.0 | 16.90 | 22.10 | 94.30 | 281.00 | 2.40 | −3.90 | .012516 | .059172 |
| (d) | 1.29 | 3.83 | 15.15 | −5.7 | −109.9 | −548.8 | 19.40 | 25.10 | 150.00 | 588.00 | −0.19 | −0.64 | .006385 | .051546 |

or, in dimensionless form,

$$A' > 1; \quad C' > 1; \quad B' > \frac{1 + C' + 2A'}{4}.$$

Fig. 23 guarantees that the stationary value $E_4$ exists within the range:

$$\beta_2 < \min(2\alpha_2, \alpha_2 + \beta_3/4),$$

while stationary value $E_5$ exists within the range:

$$\beta_2 < \min(\beta_3/2, 2\alpha_2).$$

Hence $E_4$ and $E_5$ *simultaneously* occur if $\beta_2 < \min(\beta_3/2, 2\alpha_2)$.

The evolution of surface $E(\boldsymbol{n})$ is parametrically investigated in Fig. 28, when $A'$ is kept fixed, while $B'$ and $C'$ vary independently of each other.

Finally Fig. 29 presents the directional dependence of $E(\boldsymbol{n})$ for six materials which are representative of the class under investigation.

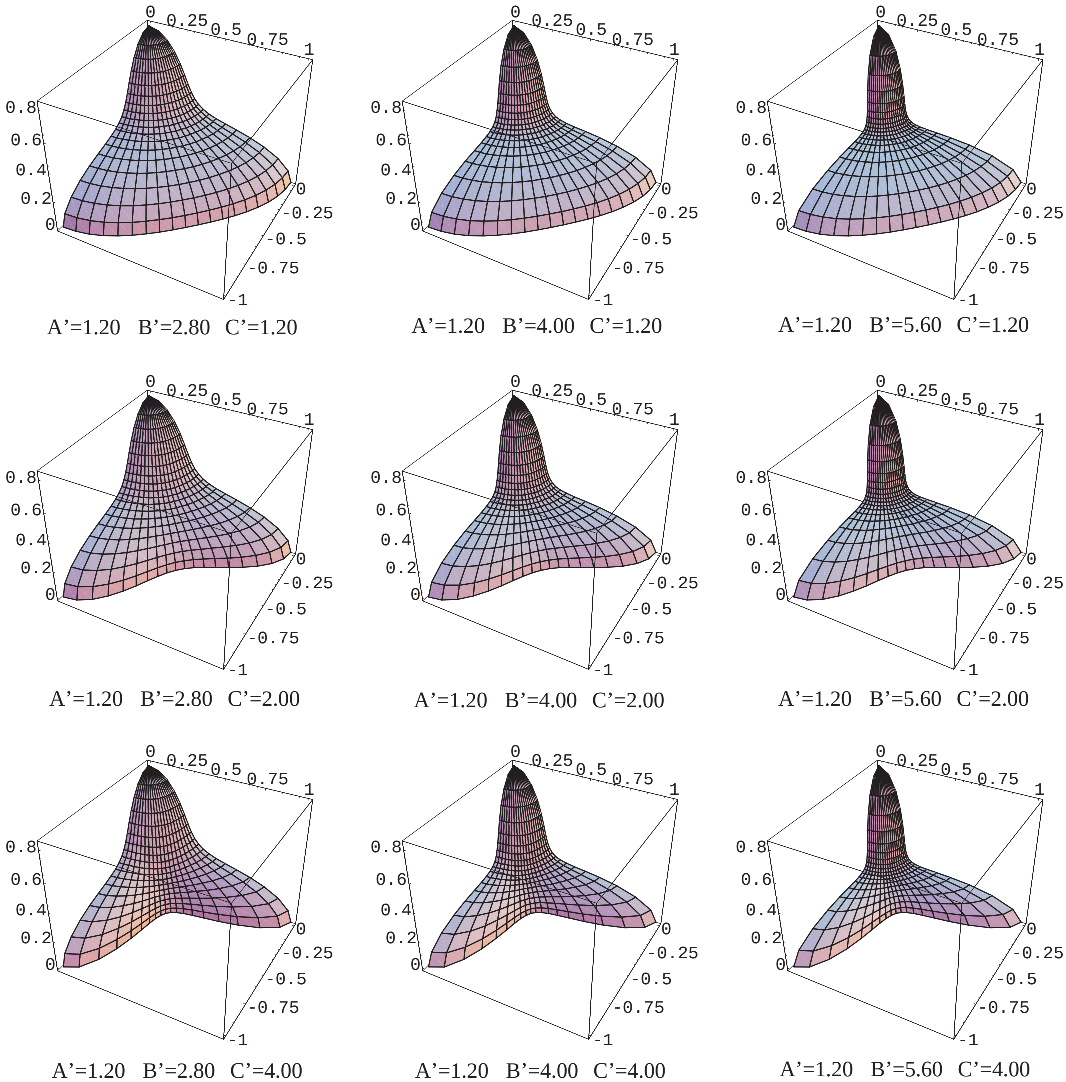


Fig. 28. Class 2f: evolution of the surface $E(\boldsymbol{n})$ as a function of changes of the dimensionless parameters $B'$ and $C'$ for a fixed value of $A'$.

## 4. Closure

For materials with tetragonal elastic symmetry, the directions along which Young's modulus attains stationary values have been analytically computed. The analytical solutions are expressed in terms of three material parameters responsible of the discrepancy from isotropy. The directions corresponding to critical values of the function $E(\boldsymbol{n})$ and the associated expressions have been discussed in detail and, in particular, 12 classes of different mechanical behaviors have been outlined. These classes cover all the possible mechanical responses in terms of Young's modulus and each class is discussed in detail. It is also shown that all these classes occur in real materials, and a wide selection of the corresponding surfaces, showing in spherical polar diagrams the directional dependence of $E(\boldsymbol{n})$, are provided as well. Future developments of the present work

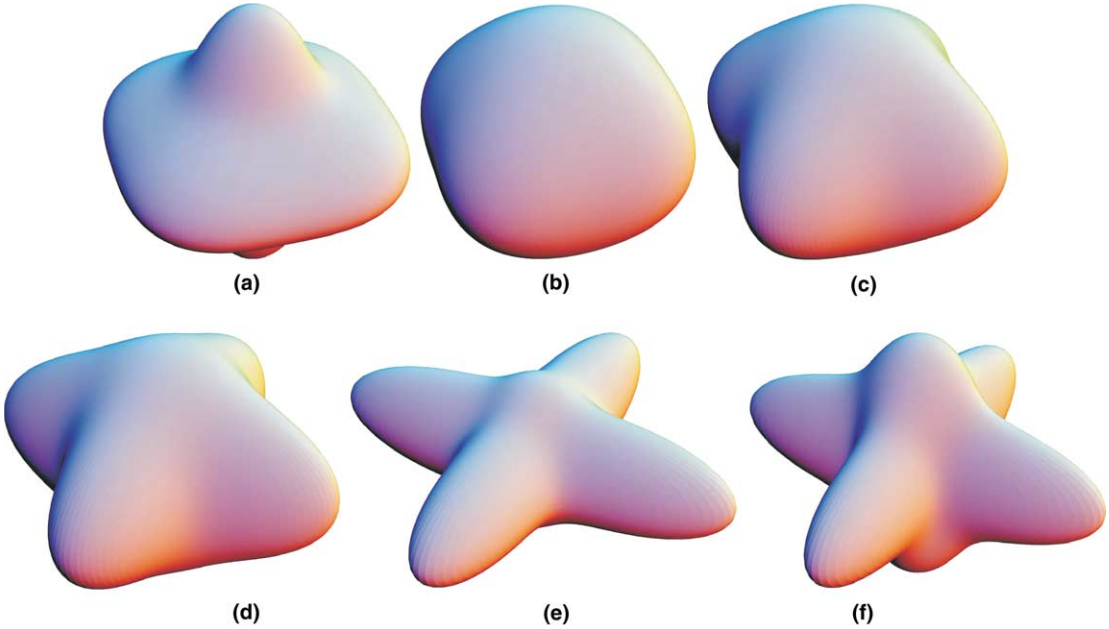

Fig. 29. Tetragonal system: materials belonging to Class 2f, defined in Table 4, and ordered for increasing values of the dimensionless material parameter $C'$. (a) $Al_2Cu$ (aluminum copper); (b) $Na_2S \cdot 9H_2O$ (sodium sulfide nonahydrate); (c) $La_{1.86}Sr_{0.14}CuO_4$ (lanthanum strontium copper oxide); (d) $K_2PtCl_4$ (potassium tetrachloroplatinate); (e) $NH_4H_2AsO_4$ (ammonium dihydrogen arsenate); (f) $KH_2AsO_4$ (potassium dihydrogen arsenate). In the following table, elastic compliance coefficients $s_{11}, \ldots, s_{13}$ (taken from Landolt and Börnstein, 1992) and material parameters $\alpha_2$, $\beta_2$, $\beta_3$ are all expressed in $TPa^{-1}$; Young's moduli $E_{min}$, $E_{max}$ are given in GPa.

| Mat. | $A'$ | $B'$ | $C'$ | $\alpha_2$ | $\beta_2$ | $\beta_3$ | $s_{11}$ | $s_{33}$ | $s_{44}$ | $s_{66}$ | $s_{12}$ | $s_{13}$ | $E_{min}$ | $E_{max}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | 1.09 | 2.09 | 1.25 | −0.6 | −16.1 | −3.70 | 7.38 | 8.01 | 35.70 | 22.40 | −1.97 | −2.40 | .084708 | .135501 |
| (b) | 1.14 | 1.20 | 1.32 | −5.3 | −14.8 | −24.0 | 37.80 | 43.10 | 114.00 | 124.00 | −12.20 | −11.80 | .022543 | .026455 |
| (c) | 1.27 | 1.38 | 1.80 | −1.2 | −3.4 | −7.2 | 4.50 | 5.70 | 14.80 | 17.20 | −0.50 | −1.20 | .158103 | .222222 |
| (d) | 1.48 | 1.64 | 2.52 | −19.2 | −51.6 | −122.0 | 40.10 | 59.30 | 165.00 | 214.00 | −5.90 | −16.60 | .014164 | .024938 |
| (e) | 2.53 | 3.11 | 4.12 | −30.2 | −83.4 | −123.6 | 19.80 | 50.00 | 149.00 | 161.00 | 1.10 | −13.00 | .017879 | .050505 |
| (f) | 1.43 | 2.54 | 4.65 | −7.1 | −50.4 | −119.8 | 16.40 | 23.50 | 93.00 | 151.00 | 0.80 | −4.90 | .021570 | .060976 |

will concern the application of the procedure here presented to weaker elastic symmetry classes, although an increasing number of elastic constants would lead to much more involved computations.

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## References

Boulanger, P., Hayes, M., 1995. On Young's modulus for anisotropic media. ASME Journal of Applied Mechanics 62, 819–820.

Cazzani, A., Rovati, M., 2003. Extrema of Young's modulus for cubic and transversely isotropic solids. International Journal of Solids and Structures 40, 1713–1744.

Cowin, S.C., Mehrabadi, M.M., 1995. Anisotropic symmetries of linear elasticity. Applied Mechanics Reviews 48 (5), 247–285.
Goens, E., 1933. Über eine verbesserte apparatur zur statischen bestimmung des drillungsmodulus von kristallstäben und ihre anwendung auf zink-einkristalle. Annalen der Physik 5 (16), 793–809.
Hayes, M., Shuvalov, A., 1988. On the extreme values of Young's modulus, the shear modulus and Poisson's ratio for cubic materials. ASME Journal of Applied Mechanics 65, 786–787.
Landolt, H.H., Börnstein, R., 1992. Numerical data and functional relationships in science and technology, III/29/a. In: Second and Higher Order Elastic Constants. Springer-Verlag, Berlin.
Nadeau, J.C., Ferrari, M., 2001. On optimal zeroth-order bounds with application to Hashin-Shtrikman bounds and anisotropy parameters. International Journal of Solids and Structures 38, 7945–7965.
Nye, J.F., 1957. Physical Properties of Crystals. Their Representation by Tensors and Matrices. Oxford University Press, Oxford.
Ostrowska-Maciejewska, J., Rychlewski, J., 2001. Generalized proper states for anisotropic elastic materials. Archives of Mechanics 53 (4–5), 501–518.
Schmid, E., Boas, W., 1935. Kristallplastizität Mit Besonderer Berücksichtigung Der Metalle. Julius Springer, Berlin.
Ting, T.C.T., 1996. Anisotropic Elasticity. Oxford University Press, New York.
Wooster, W.A., 1949. A Text-book on Crystal Physics, second ed. Cambridge University Press, Cambridge.